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Examples of nonholonomic mechanical systems [1]

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Abstract. The geometric concept of the mechanical systems subjected to general nonholonomic constraints is briefly presented, equations of motion of nonholonomic mechanical systems in the form of the reduced equations (equations of motion directly on constraint submanifold) are derived and the applications of this theory to the concrete examples of the nonholonomic mechanical systems are illustrated.

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1. Introduction

In classical mechanics one naturally encounters different kinds of constraints on the motion, mostly *holonomic* and *linear nonholonomic* and occasionally *general nonholonomic constraints.* While mechanics of holonomic systems is satisfactorily elaborated both from the physical and geometric point of view and is the standard section of courses of mechanics of particles and rigid bodies, in nonholonomic mechanics still some questions remain open.

In last decades numerous physical and engineerings applications make necessary to profound research and complete the theory of the nonholonomic systems. Therefore problems of nonholonomics mechanics are intensively studied in many recent papers, e.g. [2], [3], [5], [7], [9], [15]–[20], [22], [23] in which are used modern methods and concepts of differential geometry and global analysis and which contribute to the essential advance in both from the theoretical and application aspects.

In this paper we present geometric concept of the theory of nonholonomic mechanical systems (i.e. mechanical systems subjected to general nonholonomic constraints) developed by Krupková in 1990's ([9], [10]). Nonholonomic constraints are given by a submanifold Q of the first jet prolongation J^1Y of the configuration space Y, described by a certain system of the first order ordinary differential equations, which represents certain restriction on the positions and velocities of the moving system. This constraint submanifold is naturally endowed with the canonical distribution C and becomes a nonholonomic constraint structure. Unconstrained mechanical systems of the first order is represented by a certain family of (local) 2-forms α on the first jet prolongation J^1Y, 1-contact part of which is corresponding Euler–Lagrange form E_λ on the second jet

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prolongation J^2Y. With help of the nonholonomic constraint structure (Q, C) one can construct a new mechanical system directly on constraint submanifold Q of J^1Y. This enables one to formulate equations of constrained mechanical system in a form of the so called deformed equations (containing Lagrange multipliers), and in a form of reduced equations (equations of motion on the constraint submanifold). The obtained equations for "constraint extremals" are a particular kind of *reduced Chetaev-type equations* (i.e. with "eliminated" Lagrange multipliers), as found by Krupková [9]. They coincide with the *variational reduced equations* characterized by means of "constraint Helmholtz conditions", found by Krupková and Musilová [11], [12].

The aim of this paper is to derive, with help of basic instruments of the geometric theory, reduced equations of nonholonomic system and to apply this theory to several illustrative examples of nonholonomic mechanical systems. In each of these examples we analyse the constraints, we construct the corresponding constrained mechanical system, we derive the reduced equations of motion of this system and finally we discuss solvability of these equations. These examples show that reduced equations have the practical meaning for solution of concrete problems with nonholonomic constraints.

2. Basic structures

In this section we recall basic structures and concepts used in the calculus of variations on fibered manifolds, for more detailes we refer to [8] and [9]. Throughout the paper we consider a fibered manifold $\pi\colon Y \longrightarrow X$ with $\dim X = 1$, $\dim Y = m+1$, its jet prolongations $\pi_1\colon J^1Y \longrightarrow X$ and $\pi_2\colon J^2Y \longrightarrow X$ and the jet projections $\pi_{1,0} : J^1Y \longrightarrow Y$ and $\pi_{2,1} : J^2Y \longrightarrow J^1Y$. We denote (t, q^σ), $1 \le \sigma \le m$, fibered coordinates on Y, $(t, q^\sigma, \dot q^\sigma)$ associated coordinates on J^1Y, and by

$$\omega^\sigma = dq^\sigma - \dot q^\sigma\, dt, \quad 1 \le \sigma \le m \tag{1}$$

canonical contact 1-forms annihilating the *Cartan distribution* on J^1Y. Whenever possible, the summation convention is used. If $f(t, q^\sigma, \dot q^\sigma)$ is a function we write

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^\sigma}\dot q^\sigma + \frac{\partial f}{\partial \dot q^\sigma}\ddot q^\sigma, \qquad \frac{\bar d f}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^\sigma}\dot q^\sigma. \tag{2}$$

A (local) section δ of π_1 is called *holonomic* if $\delta = J^1\gamma$ for a section γ of π.

A vector field ξ on J^1Y is called π_1*-vertical* (or simply *vertical*) if $T\pi_1 \cdot \xi = 0$. Similarly, ξ is called $\pi_{1,0}$*- vertical* if $T\pi_{1,0} \cdot \xi = 0$. A differential form ρ is called *contact* if $J^1\gamma^*\rho = 0$ for every section γ of π. A differential form ρ is called *horizontal* if $i_\xi \rho = 0$ for every vertical vector field ξ. We denote by h the operator assigning to ρ its horizontal part. Every 2-form on J^1Y is contact and admits a *unique decomposition* $\pi_{2,1}^*\rho = \rho_1 + \rho_2$, where ρ_1 is a 1-*contact* form on J^2Y (i.e. for every vertical vector field ξ, $i_\xi\rho_1$ is a horizontal form), and ρ_2 is a 2-*contact* form (i.e. for every vertical vector field ξ, $i_\xi\rho_2$ is a 1-contact form). We denote by p_1, and p_2 operators assigning to ρ its 1-contact and 2-contact part, respectively.

By a *distribution* on J^1Y we shall mean a mapping D assigning to every point $z \in J^1Y$ a vector subspace $D(z)$ of the vector space T_zJ^1Y. A distribution is said to be of a *constant rank* if $\dim D(z)$ does not depend on z. A distribution can be spanned by a system of (local) vector fields. A distribution is called *continuous* (resp. *smooth*) if it can be spanned by a system of continuous (resp. smooth) vector fields. If D is a distribution, we denote by D^0 its annihilator, i.e., the set of all 1-forms η_κ on J^1Y such that $i_{\xi_\iota}\eta_\kappa = 0$ for every vector field ξ_ι belonging to D. In this sense, every distribution can be defined by a system of (local) 1-forms. For a distributions of a constant rank the description by means of vector fields is completely equivalent with that by means of 1-forms. If, however, rank D is not constant then a distribution which can be generated by a system of smooth 1-forms is not continuous. Recall that a section δ of π_1 is called an *integral section* of D if $\delta^*\eta = 0$ for every 1-form η belonging to D^0.

3. Unconstrained mechanical systems

A 2-form E on J^2Y is called a *dynamical form* if it is 1-contact and $\pi_{2,0}$-horizontal. This means that E is dynamical form iff in every fiber chart

(3) $$E = E_\sigma(t, q^\nu, \dot{q}^\nu, \ddot{q}^\nu)\, \omega^\sigma \wedge dt.$$

In this paper we restrict to dynamical forms with the components *affine in the second derivatives*, i.e., we suppose that the components of E are of the form

(4) $$E_\sigma = A_\sigma(t, q^\nu, \dot{q}^\nu) + B_{\sigma\rho}(t, q^\nu, \dot{q}^\nu)\ddot{q}^\rho.$$

A section γ of π is called a *path* of E if

$$E \circ J^2\gamma = 0.$$

In fiber coordinates this equation represents a system of m second-order ordinary differential equations

(5) $$A_\sigma\left(t, \gamma^\nu, \frac{d\gamma^\nu}{dt}\right) + B_{\sigma\rho}\left(t, \gamma^\nu, \frac{d\gamma^\nu}{dt}\right)\frac{d^2\gamma^\rho}{dt^2} = 0$$

for the components $\gamma^\nu(t)$, $1 \le \nu \le m$, of the section γ. The equations of this type are called the *motion equations* and its solutions are called *paths*. It is important to note that the equations (5) need not be of the form of "solvable with respect to the second derivatives". If they do posses this property, i.e. if

(6) $$\det(B_{\sigma\rho}) \neq 0,$$

then the form E is called *regular*.

An intrinsic form of equations (5) is obtained by means of certain family of equivalent local first order 2-forms. We say that a 2-form α on J^1Y is called *Lepage 2-form* associated to a dynamical form E if $p_1\alpha = E$. Immediately one gets for E the following *equivalence class* of associated Lepage 2-forms:

(7) $$[\alpha] = A_\sigma\omega^\sigma \wedge dt + B_{\sigma\nu}\omega^\sigma \wedge d\dot{q}^\nu + F$$

where F runs over the 2-contact 2-forms of order one, $F = F_{\sigma\nu}(t, q^\rho, \dot{q}^\rho)\, \omega^\sigma \wedge \omega^\nu$. This equivalence class is then called a *first-order mechanical system* on π.

For a Lepagean 2-form α put

$$\Delta^0_\alpha = \operatorname{span}\{i_\xi\alpha\},$$

where ξ runs over the set of all π_1-vertical vector fields on J^1Y. The corresponding distribution Δ_α is a distribution on J^1Y, will be called the *dynamical distribution* of α. One can easily see that the dynamical distribution of α can be generated by means of the following 1-forms:

$$A_\sigma\, dt + 2F_{\sigma\nu}\, \omega^\nu + B_{\sigma\nu}\, d\dot{q}^\nu, \qquad B_{\sigma\nu}\, \omega^\nu.$$

Notice that this distribution need not be of a constant rank, and that at each point of J^1Y one has $\operatorname{rank}\Delta_\alpha \ge 1$.

The following is an important property of Lepagean 2-forms.

Proposition 1. *Let E be a dynamical form.*

(1) *Let α_1, α_2 be equivalent Lepagean 2-forms associated to E, let $\Delta_{\alpha_1}, \Delta_{\alpha_2}$ be the corresponding dynamical distributions. Then the holonomic integral sections of Δ_{α_1} and Δ_{α_2} coincide.*

(2) *The set of holonomic integral sections of any dynamical distribution of E coincides with the set of paths of E.*

(3) *A section γ of π is a path of a dynamical form E if and only if for every π_1-vertical vector field ξ on J^1Y*

$$J^1\gamma^* \, i_\xi \alpha = 0, \tag{8}$$

where α is a Lepagean 2-form associated to E.

A mechanical system will be denoted by $[\alpha]$, and the class of the corresponding dynamical distributions will be denoted by $[\Delta_\alpha]$. If there is not danger we shall use the notations α, and Δ_α, respectively, where α (resp., Δ_α) is a respresentative of the class $[\alpha]$ (resp., $[\Delta_\alpha]$).

A mechanical system $[\alpha]$ will be called *regular* if there exists a dynamical distribution $\Delta_\alpha \in [\Delta_\alpha]$ such that $\mathrm{rank}\Delta_\alpha = 1$ on J^1Y. We obtain the following useful characterization of regular mechanical systems:

Proposition 2. *Let $[\alpha]$ be the mechanical system related to a dynamical form $E = (A_\sigma + B_{\sigma\nu}\ddot{q}^\nu)\omega^\sigma \wedge dt$, and let $[\Delta_\alpha]$ be the corresponding class of dynamical distributions. The following conditions are equivalent:*

(1) *The mechanical system is regular.*

(2) *The matrix $(B_{\sigma\nu})$ is everywhere regular.*

(3) *Each of the dynamical distributions belonging to $[\Delta_\alpha]$ has rank one.*

(4) *All the dynamical distributions belonging to $[\Delta_\alpha]$ coincide and*

$$\Delta_\alpha = \mathrm{span}\left\{ \frac{\partial}{\partial t} + \dot{q}^\sigma \frac{\partial}{\partial q^\sigma} - B^{\sigma\rho} A_\rho \frac{\partial}{\partial \dot{q}^\sigma} \right\},$$

$$\Delta^0_\alpha = \mathrm{span}\{A_\sigma \, dt + B_{\sigma\nu} \, d\dot{q}^\nu, \, \omega^\sigma, \, 1 \le \sigma \le m\},$$

where $(B^{\sigma\nu})$ is the inverse matrix to $(B_{\sigma\nu})$.

(5) *The equations for paths of E have an equivalent form*

$$\ddot{q}^\sigma = -B^{\sigma\rho} A_\rho, \quad 1 \le \sigma \le m$$

along $J^2\gamma$.

Consider now the particular case when a dynamical form E is *locally variational*, i.e., in a neighborhood of every point of J^1Y there exists a Lagrange function $L(t, q^\sigma, \dot{q}^\sigma)$ such that the components $E_\sigma = A_\sigma + B_{\sigma\rho}\ddot{q}^\rho$ have the form

$$E_\sigma(L) = \frac{\partial L}{\partial q^\sigma} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\sigma}, \tag{9}$$

where

$$A_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{\partial^2 L}{\partial t \partial \dot{q}^\sigma} - \frac{\partial^2 L}{\partial q^\nu \partial \dot{q}^\sigma}\dot{q}^\nu, \quad B_{\sigma\nu} = -\frac{\partial^2 L}{\partial \dot{q}^\sigma \partial \dot{q}^\nu}. \tag{10}$$

Then the equivalence class $[\alpha]$ of Lepagean 2-form of E contains a *unique closed* 2-form α_E, called the *Lepagean equivalent* of E. Conversely, if the equivalence class $[\alpha]$ of Lepagean 2-forms of E contains a *closed* 2-form then E is locally variational. A mechanical system related with a locally variational form will be called a *Lagrangian system.*

Remark 1. Notice that the concept of a Lepagean 2-form is a generalization of the concept of *symplectic form* to the general situation (time-dependent, higher-order, not necessarily regular).

If $\lambda = L\,dt$ is a Lagrangian on J^1Y, then corresponding dynamical form E_λ is called its *Euler–Lagrange form.* We denote by θ_λ its *Lepage equivalent* or *Cartan form.* In fibered coordinates where $\lambda = Ldt$ we have

$$\theta_\lambda = Ldt + \frac{\partial L}{\partial \dot q^\sigma}\omega^\sigma, \qquad E_\lambda = E_\sigma(L)\omega^\sigma \wedge dt, \tag{11}$$

where components $E_\sigma(L)$ are *Euler–Lagrange expressions* (9).
If $[\alpha]$ is a Lagrangian system, then arbitrary representative α of this class is of the form $\alpha = d\theta_\lambda + F$, where F runs over all $\pi_{1,0}$-horizontal 2-contact 2-forms. The corresponding motion equations (8) of the mechanical system $[\alpha]$, which are called in this case *Euler–Lagrange equations* of λ take the form

$$J^1\gamma^* i_\xi d\theta_\lambda = 0, \tag{12}$$

where ξ is a π_1-vertical vector field on J^1Y. Using identity $p_1\alpha = E_\lambda$ we can write these equations in the standard way $E_\sigma(L) \circ J^2\gamma = 0$. Finally, the regularity condition (6) takes the familiar form

$$\det\left(\frac{\partial^2 L}{\partial \dot q^\sigma \partial \dot q^\nu}\right) \neq 0. \tag{13}$$

4. The nonholonomic constraint structure

In this section we shall introduce the concept of a system of nonholonomic constraints, and of a related constraint force. Constraints will be interpreted as a certain submanifold of J^1Y canonically endowed with a distribution.

Let k be an integer. By a *constraint submanifold* in J^1Y we mean a fibered submanifold $\pi_{1,0}|_Q : Q \longrightarrow Y$ of the fibered manifold $\pi_{1,0} : J^1Y \longrightarrow Y$. We denote by ι the canonical embedding of Q into J^1Y, and suppose $\mathrm{codim} Q = k < m$. (cf. for example [9], [10], [17], [22], [23]). Locally, Q can be given by equations

$$f^i = 0, \quad 1 \le i \le k, \quad \text{where} \quad \mathrm{rank}\left(\frac{\partial f^i}{\partial \dot q^\sigma}\right) = k, \tag{14}$$

or, equivalently in an explicit form

$$\dot q^{m-k+i} = g^i(t, q^\sigma, \dot q^1, \dot q^2, \dots, \dot q^{m-k}), \quad 1 \le i \le k, \tag{15}$$

called a *system of k nonholonomic constraints.*

The presence of a constraint submanifold in J^1Y gives rise to a concept of a *constrained section* as a local section $\bar\delta$ of the fibered manifold π_1 such that for every $x \in \mathrm{dom}\bar\delta$: $\bar\delta(x) \in Q$, and a *Q-admissible* section as a section $\bar\gamma$ of the fibered manifold π such that $J^1\bar\gamma(x) \in Q$ for every $x \in \mathrm{dom}\bar\gamma$. The set of all Q-admissible sections $\bar\gamma$ of π will be denoted by $\bar\Gamma^Q(\pi)$.

The submanifold Q is naturally endowed with a distribution, called the *canonical distribution* [9], or *Chetaev bundle* [17], and denoted by C. It is annihilated by a system of k linearly independent (local) 1-forms

$$\varphi^i = \iota^* \phi^i, \quad \text{where} \quad \phi^i = f^i dt + \frac{\partial f^i}{\partial \dot{q}^\sigma} \omega^\sigma, \quad 1 \leq i \leq k, \tag{16}$$

i.e.

$$\varphi^i = -\sum_{l=1}^{m-k} \frac{\partial g^i}{\partial \dot{q}^l} \omega^l + \iota^* \omega^{m-k+i}, \quad 1 \leq i \leq k, \tag{17}$$

called *canonical constraint* 1*-forms.* The ideal in the exterior algebra of forms on Q generated by the annihilator of C is called the *constraint ideal*, and denoted by $I(C^0)$, or simply I; its elements are called *constraint forms.* The pair (Q, C) is then called a *(nonholonomic) constraint structure* on the fibered manifold π [9], [10].

Remark 2. Notice that the holonomic integral sections of the canonical distribution C coincide with the holonomic constrained sections of π_1, i.e. have the form $\bar{\delta} = J^1\bar{\gamma}$, where $\bar{\gamma}$ is a Q-admissible section of π.

Remark 3. From the point of view of physics, the rank of the canonical distribution C has the meaning of the number of (generalized, or "phase space") *degrees of freedom* of systems constrained to Q, and the canonical distribution itself represents *possible (generalized) displacements.* Its π_1- and $\pi_{1,0}$-vertical subdistribution then has the meaning of *virtual (generalized) displacements* and *virtual velocities*, respectively.

Let $\widetilde{Q}$ be the *lift* of Q in J^2Y, i.e. the manifold of all points $J^2_x\gamma \in J^2Y$ such that $J^1_x\gamma \in Q$. If Q is given by (15) then equations of $\widetilde{Q}$ are

$$\dot{q}^{m-k+i} = g^i(t, q^\sigma, \dot{q}^1, \dot{q}^2, \ldots, \dot{q}^{m-k}), \quad \ddot{q}^{m-k+i} = \frac{dg^i}{dt}. \tag{18}$$

We denote by $\rho : \widetilde{Q} \longrightarrow Q$ the corresponding jet projection (i.e. $\rho = \pi_{2,1}|_{\widetilde{Q}}$). The distribution $\widetilde{C}$ on $\widetilde{Q}$, such that for every $y \in \widetilde{Q}$

$$T_y\rho(\widetilde{C}(y)) = C(\rho(y)) \tag{19}$$

is called the *lift of the canonical distribution* C [13]. Since rank$\widetilde{C} = 3m + 1 - 3k$, and dim$\widetilde{Q} = 3m + 1 - 2k$ one has corank$\widetilde{C}$ = corank$C = k$. The annihilator of $\widetilde{C}$ is locally spanned by 1-forms $\tilde{\varphi}^i = \rho^*\varphi^i$, $1 \leq i \leq k$. But since canonical constraint 1-forms are $\pi_{1,0}$-horizontal, $\rho^*\varphi^i = \varphi^i$, and thus $\widetilde{C}^0 = C^0 = \text{span}\{\varphi^i\}$. We denote by $\tilde{I}$ the ideal on $\widetilde{Q}$, generated by $\widetilde{C}^0$.

5. Constrained systems

Let E is a dynamical form and $[\alpha]$ corresponding mechanical system. With help of the nonholonomic constraint structure (Q, C) one can construct a new mechanical system directly on constraint submanifold Q of J^1Y. By a *(non-holonomic) constrained system* related with $[\alpha]$ and the constraint structure (Q, C) we mean the equivalence class $[\alpha_Q]$ on Q, where

$$\alpha_Q = \iota^*\alpha + F + \varphi_{(2)}, \tag{20}$$

where F is a 2-contact 2-form and $\varphi_{(2)}$ is a constraint 2-form ([9], [10]). Computing the coordinate expression, we get that a representative of the class $[\alpha_Q]$ takes the form

$$\alpha_Q = \sum_{l=1}^{m-k} A'_l \omega^l \wedge dt + \sum_{l,s=1}^{m-k} B'_{l,s} \omega^l \wedge d\dot{q}^s + F + \varphi_{(2)}, \tag{21}$$

where components A'_l , $B'_{l,s}$ are determined by relations

$$A'_l = \left(A_l + \sum_{i=1}^{k} A_{m-k+i} \frac{\partial g^i}{\partial \dot{q}^l} + \sum_{j=1}^{k} \left(B_{l,m-k+j} + \sum_{i=1}^{k} B_{m-k+i,m-k+j} \frac{\partial g^i}{\partial \dot{q}^l} \right) \frac{\bar{d} g^j}{dt} \right)_\iota,$$

$$B'_{l,s} = \left(B_{l,s} + \sum_{i=1}^{k} \left(B_{l,m-k+i} \frac{\partial g^i}{\partial \dot{q}^s} + B_{m-k+i,s} \frac{\partial g^i}{\partial \dot{q}^l} \right) + \sum_{i,j=1}^{k} B_{m-k+i,m-k+j} \frac{\partial g^j}{\partial \dot{q}^l} \frac{\partial g^i}{\partial \dot{q}^s} \right)_\iota.$$

Equations of motion of the constrained system $[\alpha_Q]$, then have the following intrinsic form:

$$J^1\gamma^* i_\xi \alpha_Q = 0 \quad \text{for every vector field } \xi \in C, \tag{22}$$

where α_Q is any 2-form belonging to the the class $[\alpha_Q]$. In fibered coordinates,

$$f^i \circ J^1\gamma = 0, \quad \left(A'_l + \sum_{s=1}^{m-k} B'_{l,s} \ddot{q}^s \right) \circ J^2\gamma = 0. \tag{23}$$

Notice that the above system of equations can be viewed as 2nd order equations for $\gamma^1, \gamma^2, ..., \gamma^{m-k}$ dependent on t and the parameters $q^{m-k+1}, q^{m-k+2}, ..., q^m$, which have to be determined as functions $\gamma^{m-k+1}(t), \gamma^{m-k+2}(t), ..., \gamma^m(t)$ from the equations of the constraint. The first order ODEs of the constraint simply mean reduction the set of all sections of the fibered manifold π only on the subset of Q-admissible sections $\bar{\gamma}$.

Naturally, a *constrained dynamical distribution* Δ_{α_Q} associated to the 2-form α_Q will be defined as a subdistribution of the canonical distribution C, generated by means of the 1-forms $i_\xi \alpha_Q$, where ξ runs over all π_1-vertical vector fields on Q. Hence, the annihilator of Δ_{α_Q} is spanned by the 1-forms

$$\iota^*\varphi^i, \quad A'_l\, dt + 2F'_{ls}\, \omega^s + B'_{ls}\, d\dot{q}^s, \quad B'_{ls}\, \omega^s,$$

where $1 \le l \le m - k$.

The equivalence class $[\alpha_Q]$ naturally gives rise to the class $[\Delta_{\alpha_Q}]$ of associated constrained distributions.

We can see that for all $\Delta_{\alpha_Q} \in [\Delta_{\alpha_Q}]$, $\operatorname{rank} \Delta_{\alpha_Q} \ge 1$. In analogy with the unconstrained case, the constrained system will be called *regular* on an open set $V \subset Q$ if for some constrained dynamical distributionΔ_{α_Q} on V belonging to the class $[\Delta_{\alpha_Q}]$ corresponding to $[\alpha]$, $\operatorname{rank} \Delta_{\alpha_Q} = 1$ on V.

In keeping with Proposition 2 we can see that for a regular constrained mechanical system we have following.

Proposition 3. *Let $[\alpha_Q]$ be the constrained mechanical system related to a mechanical system $[\alpha]$, and let $[\Delta_{\alpha_Q}]$ be the corresponding equivalence class of constrained dynamical distributions. Let $V \subset Q$ be an open set. The following conditions are equivalent:*

(1) *The constrained mechanical system $[\alpha_Q]$ is regular on V.*

(2) *The regularity condition*

$$\det(B'_{ls}) \ne 0 \quad 1 \le l, s \le m - k \tag{24}$$

is satisfied at each point of V.

(3) *Each of the dynamical distributions of* $[\Delta_{\alpha_Q}]$ *has rank one on* V.

(4) *All the dynamical distributions of* $[\alpha_Q]$ *on* V *coincides; their annihilator is spanned by* 1*-forms*

$$\iota^*\varphi^i, \quad A'_l\, dt + B'_{ls}\, d\dot{q}^s, \quad \omega^l, \quad 1 \le i \le k, \quad 1 \le l \le m-k.$$

(5) *The constrained equations of motion have an equivalent form*

$$\ddot{q}^l = -B'^{ls} A'_s, \quad 1 \le l \le m-k,$$

$$f^i = 0, \quad 1 \le i \le k,$$

where (B'^{ls}) *is the inverse matrix to* (B'_{ls}).

If the constraint Q is defined by equations in an explicit form then the generators of $\Delta^0_{\alpha_Q}$ for a regular system $[\alpha_Q]$ take the following form:

$$dq^{m-k+i} - g^i dt, \quad A'_l\, dt + B'_{ls}\, d\dot{q}^s, \quad \omega^l, \quad 1 \le i \le k, \quad 1 \le l \le m-k,$$

and the constrained equations of motion become

$$\ddot{q}^l = -B'^{ls} A'_s, \quad 1 \le l \le m-k,$$

$$\dot{q}^{m-k+i} j = g^i, \quad 1 \le i \le k.$$

Notice that a constrained system of a regular (resp. singular) mechanical system need not be regular (resp. singular).

6. Constrained Lagrangian systems

Let $[\alpha_Q]$ be a constrained system related with unconstrained system $[\alpha]$ and the constraint structure (Q, C). The 2-form α_Q need not be closed and moreover, in general, there is no closed 2-form in the class $[\alpha_Q]$.

Proposition 4. *A constrained system* $[\alpha_Q]$ *is Lagrangian if and only if there is a closed* 2*-form belonging to the class* $[\alpha_Q]$.

Consider now the particular case when an unconstrained system $[\alpha]$ is Lagrangian system. This means that there exists a 2-form $\alpha_E \in [\alpha]$ such that $d\alpha_E = 0$. A nonholonomic constraint $Q \subset J^1Y$ being given, the 2-form $(\alpha_E)_Q$ need not be closed. Consequently, a constrained system arising from a Lagrangian system need not be Lagrangian.

Proposition 5. *A constrained mechanical system* $[(\alpha_E)_Q]$ *arising from a Lagrangian system* $[\alpha_E]$ *is a Lagrangian if and only if there exists a* 2*-contact* $\pi_{1,0}$*-horizontal* 2*-form* F *on* Q *and constraint* 2*-form* β *such that* $d(F + \beta) = 0$.

If $\lambda = Ldt$ is a (possible local) Lagrangian for E and θ_λ its Cartan form then (locally) $\alpha_E = d\theta_\lambda$ and $[\alpha] = [d\theta_\lambda]$ is the corresponding unconstrained Lagrangian system. By a *(non-holonomic) constrained system arising from the Lagrangian system* $[\alpha]$ and the constraint structure (Q, C) we mean the equivalence class $[\alpha_Q]$ on Q, where

$$\alpha_Q = \iota^* d\theta_\lambda + F + \varphi_{(2)}, \tag{25}$$

where F is a 2-contact 2-form and $\varphi_{(2)}$ is a constraint 2-form.

Motion equations (22) become

$$J^1\gamma^* i_\xi \iota^* d\theta_\lambda = 0 \quad \text{for every vector field } \xi \in C. \tag{26}$$

Let us find an explicit expression of the motion equation of the constrained system $[(d\theta_\lambda)_Q]$ arising from the Lagrangian system. In fibered coordinates they have the form (23). For calculation of components A'_l, $B'_{l,s}$ by means of L we use coefficients A_l, A_{m-k+i} and $B_{l,s}$, $B_{l,m-k+i}$, $B_{m-k+i,m-k+j}$ of Euler-Lagrange expressions in (9), which are given by (10) substituting the corresponding indexes for σ, ν and using equations (15) of the constraint submanifold Q. Introducing Lagrange function $\bar{L}$ on the constraint submanifold Q as the restriction of the original unconstrained Lagrange function L on Q, i.e. $\bar{L} = L \circ \iota$, thus $\bar{L}(t, q^\sigma, \dot{q}^l) = L(t, q^\sigma, \dot{q}^l, g^i(t, q^\sigma, \dot{q}^l))$ and after several arrangements with using identities between first and second order derivatives of L and $\bar{L}$ with respect to corresponding variables (see [13] for more detailes) we obtain final expression for components A'_l :

$$
\begin{aligned}
A'_l = {} & \frac{\partial \bar{L}}{\partial q^l} + \frac{\partial \bar{L}}{\partial q^{m-k+i}} \frac{\partial g^i}{\partial \dot{q}^l} - \frac{\bar{d}}{\bar{d}t} \frac{\partial \bar{L}}{\partial \dot{q}^l} + \\
& + \left(\frac{\partial L}{\partial \dot{q}^{m-k+j}} \right)_\iota \left[\frac{\bar{d}}{\bar{d}t} \left(\frac{\partial g^j}{\partial \dot{q}^l} \right) - \frac{\partial g^j}{\partial q^l} - \frac{\partial g^j}{\partial q^{m-k+i}} \frac{\partial g^i}{\partial \dot{q}^l} \right]
\end{aligned} \tag{27}
$$

and for $B'_{l,s}$

$$
B'_{l,s} = -\frac{\partial^2 \bar{L}}{\partial \dot{q}^l \partial \dot{q}^s} + \left(\frac{\partial L}{\partial \dot{q}^{m-k+i}} \right)_\iota \frac{\partial^2 g^i}{\partial \dot{q}^l \partial \dot{q}^s}. \tag{28}
$$

Now constraint equations of motion (23) take the form

$$
\begin{aligned}
& \frac{\partial \bar{L}}{\partial q^l} + \frac{\partial \bar{L}}{\partial q^{m-k+i}} \frac{\partial g^i}{\partial \dot{q}^l} - \frac{d}{dt} \left(\frac{\partial \bar{L}}{\partial \dot{q}^l} \right) + \\
& + \left(\frac{\partial L}{\partial \dot{q}^{m-k+j}} \right)_\iota \left[\frac{d}{dt} \left(\frac{\partial g^j}{\partial \dot{q}^l} \right) - \frac{\partial g^j}{\partial q^l} - \frac{\partial g^j}{\partial q^{m-k+i}} \frac{\partial g^i}{\partial \dot{q}^l} \right] = 0.
\end{aligned} \tag{29}
$$

Rewriting the *regularity condition* (24) by means of constrained Lagrangian $\bar{\lambda} = \bar{L} dt$ we obtain

$$
\det \left(\frac{\partial \bar{L}}{\partial \dot{q}^l \partial \dot{q}^s} - \left(\frac{\partial L}{\partial \dot{q}^{m-k+i}} \right)_\iota \frac{\partial^2 g^i}{\partial \dot{q}^l \partial \dot{q}^s} \right) \neq 0. \tag{30}
$$

We have an important relation

$$
\iota^* \theta_\lambda - \theta_{\iota^* \lambda} \in I(C^0).
$$

Consequently $d\theta_{\iota^*\lambda}$ belongs to the class $[\iota^* d\theta_\lambda]$ if and only if the constraint ideal I is diferential (i.e. $d\varphi \in I$ for arbitrary constraint form φ) and the constraint ideal I is diferential if and only if the constraint is semi-holonomic. Hence, in the semi-holonomic case the motion equations of the constraint system arising from the Lagrangian system have an equivalent form

$$
J^1 \gamma^* i_\xi d\theta_{\iota^* \lambda} = 0
$$

for every vector fields $\xi \in \mathcal{C}$.

Finally we can pronounce the conclusion, that *the pull-back of an unconstrained Lagrangian λ to a constraint submanifold has the meaning of a Lagrangian for the constrained system if and only if the constraint is semi-holonomic.* In the case of more complicated constraints (linear nonintegrable, non-linear in velocities) the constrained Lagrangian system cannot be determined by a *single function*, but by an *equivalence class of 1-forms.*

7. Illustrative examples

Example 1. (see [1], pp. 234-235)
Consider a "free particle" on $\mathbb{R}^2$ which moves along curve angular coefficient of which is proportional to time passed from the begening of the motion.

We denote by (t) the coordinate on $X = \mathbb{R}$, by (t, x, y) the fiber coordinates on $Y = \mathbb{R} \times \mathbb{R}^2$ and associated coordinates $(t, x, y, \dot{x}, \dot{y})$ on $J^1Y = \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$.

The Lagrangian of this problem has the simple form

$$\lambda = L\,dt = \tfrac{1}{2}\,(\dot{x}^2 + \dot{y}^2)\,dt \tag{31}$$

and we consider a first order mechanical system $[\alpha]$ on the fibered manifold $\mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$, related with a dynamical form

$$E = (A_\sigma + B_{\sigma\rho}\ddot{q}^\rho)\,dq^\sigma \wedge dt = -m\ddot{x}\,dx \wedge dt - m\ddot{y}\,dy \wedge dt, \tag{32}$$

where components A_σ , $B_{\sigma\rho}$ given by (10) are in this case $A_\sigma = 0$, $B_{\sigma\rho} = -m\,\delta_{\sigma\rho}$, $\sigma, \rho = 1, 2$. Associated Lepage 2-form α (7) is then expressed by

$$\alpha = -m\,\omega^1 \wedge d\dot{x} - m\,\omega^2 \wedge d\dot{y}, \tag{33}$$

where $\omega^1 = dx - \dot{x}\,dt$, $\omega^2 = dy - \dot{y}\,dt$.

The motion of the mechanical system $[\alpha]$ is for $t > 0$ subjected to one nonholonomic constraint Q given by equation

$$f(t, x, y, \dot{x}, \dot{y}) \equiv kt\dot{x} - \dot{y} = 0, \tag{34}$$

or equivalently in normal form $\dot{y} = g(t, x, y, \dot{x}) = kt\dot{x}$, where k is a positive constant. This equation defines constraint submanifold $Q \subset J^1Y$ since condition (14) is evidently satisfied. The constraint 1-form (16) takes the form

$$\varphi = kt\,dx - dy. \tag{35}$$

The constrained system $[\alpha_Q]$ related to the mechanical system $[\alpha]$ and the constraint Q is the equiavalence class of the 2-form

$$\alpha_Q = A'_1\,\omega^1 \wedge dt + B'_{11}\,\omega^1 \wedge d\dot{x} + F + \varphi_{(2)} \tag{36}$$

on Q, where

$$A'_1 = -mk^2t\dot{x}, \qquad B'_{11} = -m(1 + k^2t^2)$$

and F is any 2-contact 2-form and $\varphi_{(2)}$ is any constraint 2-form. The matrix (B'_{ls}) is only the number $(-m(1 + k^2t^2))$, which is always nonzero, it means, that the constrained system $[\alpha_Q]$ is always regular.

The equation of motion of the constrained system (reduced equation) is

$$[mk^2t\dot{x} + m(1 + k^2t^2)\ddot{x}] \circ J^2\bar{\gamma} = 0 \tag{37}$$

for the Q-admissible sections $\bar{\gamma} = (t, x(t), y(t))$ (i.e. for the sections satisfying the constraint equation $f \circ J^1\gamma = 0$). Equation (37) is easy solvable: using substitution $u = \dot{x}$ and after separation of variables we obtain

$$u(t) = \dot{x}(t) = \frac{v_0}{\sqrt{1 + k^2t^2}}, \tag{38}$$

and substituting to the equation (34) of the constraint

$$\dot{y}(t) = \frac{kv_0\, t}{\sqrt{1+k^2t^2}}. \tag{39}$$

Integrating (38) and (39) we get general solution of the considered problem in the parametric form:

$$x(t) = x_0 + \frac{v_0}{k} \ln(kt + \sqrt{1+k^2t^2}\,),$$

$$y(t) = y_0 + \frac{v_0}{k}(kt + \sqrt{1+k^2t^2}\,),$$

where v_0 is the initial velocity and x_0, y_0 are coordinates of the initial position of the particle.

Example 2. (See [9], pp. 5123, Example 1)
Consider a "free particle" on $\mathbb{R}^3$ which moves such a way, that square of the magnitude of the instantaneous velocity decreases proportional to reciprocal value of time passed from the begening of the motion.

We denote by (t) the coordinate on $X = \mathbb{R}$, by (t, q^1, q^2, q^3) the fiber coordinates on $Y = \mathbb{R} \times \mathbb{R}^3$ and associated coordinates $(t, q^1, q^2, q^3, \dot{q}^1, \dot{q}^2, \dot{q}^3)$ on $J^1Y = \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$.

Let us consider a first order mechanical system $[\alpha]$

$$\alpha = \, - m(\omega^1 \wedge d\dot{q}^1 + \omega^2 \wedge d\dot{q}^2 + \omega^3 \wedge d\dot{q}^3) + F$$

on the fibered manifold $\mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{R}$, related with a dynamical form

$$E = \sum_{\sigma=1}^{3} m\ddot{q}^\sigma \, dq^\sigma \wedge dt. \tag{40}$$

The motion of the mechanical system $[\alpha]$ is for $t > 0$ subject to one nonholonomic constraint Q given by equation

$$f(t, q^\sigma, \dot{q}^\sigma) \equiv t[(\dot{q}^1)^2 + (\dot{q}^2)^2 + (\dot{q}^3)^2] - 1 = 0, \tag{41}$$

which means that the magnitude of the velocity decreases proportional to $(1/\sqrt{t}\,)$. In a neighborhood of the submanifold Q

$$\operatorname{rank}\left(\frac{\partial f^i}{\partial \dot{q}^\sigma}\right) = 2t(\dot{q}^1, \dot{q}^2, \dot{q}^3) = 1, \tag{42}$$

i.e. the condition (14) is satisfied.

Let $U \subset J^1Y$ be the set of all points where $\dot{q}^3 > 0$ and consider on U canonical coordinates and the adapted coordinates $(t, q^1, q^2, q^3, \dot{q}^1, \dot{q}^2, \bar{f})$, where $\bar{f} = \dot{q}^3 - g$, $g = \sqrt{1/t - (\dot{q}^1)^2 - (\dot{q}^2)^2}$ is equation of the constraint (41) in normal form. Notice that $g > 0$ on U.

The constrained system $[\alpha_Q]$ related to the mechanical system $[\alpha]$ and the constraint Q is the equiavalence class of the 2-form

$$\alpha_Q = \sum_{l=1,2} A'_l \, \omega^l \wedge dt + \sum_{l,s=1,2} B'_{ls} \, \omega^l \wedge d\dot{q}^s + F + \varphi_{(2)} \tag{43}$$

on Q, where

$$A'_l = \left[\frac{m\dot{q}^l}{2t(\dot{q}^3)^2}((\dot{q}^1)^2 + (\dot{q}^2)^2 + (\dot{q}^3)^2)\right]_\iota = \frac{m\dot{q}^l}{2t^2g^2} \quad 1 \le l \le 2,$$

$$B'_{ls} = \left[m\left(\delta_{ls} + \frac{\dot{q}^l\dot{q}^s}{(\dot{q}^3)^2}\right)\right]_\iota = m\left(\delta_{ls} + \frac{\dot{q}^l\dot{q}^s}{g^2}\right) \quad 1 \le l, s \le 2,$$

and F is any 2-contact 2-form and $\varphi_{(2)}$ is any constraint 2-form. The matrix (B'_{ls}) is on $Q \cap U$ equivalent to the matrix

$$\begin{pmatrix} g^2 + (\dot{q}^1)^2 & \dot{q}^1\dot{q}^2 \\ \dot{q}^1\dot{q}^2 & g^2 + (\dot{q}^2)^2 \end{pmatrix}$$

hence

$$\begin{pmatrix} g^2 + (\dot{q}^1)^2 & \dot{q}^1\dot{q}^2 \\ 0 & \dfrac{g^2}{t} \end{pmatrix}$$

which is obviously regular at each point of $Q \cap U$. This means that the constrained system $[\alpha_Q]$ is regular on $Q \cap U$.

The reduced equations of motion of the constrained system are the following

$$\begin{aligned} &\left[\frac{m\dot{q}^1}{2t^2g^2} + m\left(1 + \frac{(\dot{q}^1)^2}{g^2}\right)\ddot{q}^1 + m\frac{\dot{q}^1\dot{q}^2}{g^2}\ddot{q}^2\right] \circ J^2\bar{\gamma} = 0, \\ &\left[\frac{m\dot{q}^2}{2t^2g^2} + m\left(1 + \frac{(\dot{q}^2)^2}{g^2}\right)\ddot{q}^2 + m\frac{\dot{q}^1\dot{q}^2}{g^2}\ddot{q}^1\right] \circ J^2\bar{\gamma} = 0, \end{aligned} \tag{44}$$

for the Q-admissible sections $\bar{\gamma} = (t, q^1(t), q^2(t), q^3(t))$ (i.e. for the sections satisfying the constraint equation $f \circ J^1\gamma = 0$). After arrangements we obtain equations of motion of the constrained system in the simple form:

$$\begin{aligned} \ddot{q}^1(t) &= -\frac{1}{2t}\dot{q}^1(t), \\ \ddot{q}^2(t) &= -\frac{1}{2t}\dot{q}^2(t), \\ \dot{q}^3(t) &= \sqrt{\frac{1}{t} - (\dot{q}^1)^2 - (\dot{q}^2)^2}. \end{aligned} \tag{45}$$

Solution of these equations is

$$\begin{aligned} q^1(t) &= C^1_1\sqrt{t}, + C^1_2 \\ q^2(t) &= C^2_1\sqrt{t}, + C^2_2 \\ q^3(t) &= C^3_1\sqrt{t}, + C^3_2, \end{aligned} \tag{46}$$

where C^i_j are constants connecting by the relation $C^3_1 = \sqrt{4 - (C^1_1)^2 + (C^2_1)^2}$. Analogous results are obtained if one considers the other adapted charts belonging to an atlas covering Q.

Example 3. (see [16], pp.991, Example 4.2.) *Consider a particle of the mass m moving in homogenous gravitational field (gravitational acceleration G) such a way, that square of the magnitude of the instantaneous velocity is constant at each moment.*

The configuration space is the same as in previous example, i. e. $Y = \mathbb{R} \times \mathbb{R}^3$,

$(t, q^\sigma), 1 \leq \sigma \leq 3,$ are fiber coordinates on Y. Lagrangian of this problem have the form

$$\lambda = L\,dt = [\tfrac{1}{2}\, m((\dot q^1)^2 + (\dot q^2)^2 + (\dot q^3)^2) - mGq^3]dt. \tag{47}$$

In this case mechanical system $[\alpha]$ is represented by the Lepage 2-form

$$\alpha = \; - mG\,\omega^3 \wedge dt - m(\omega^1 \wedge d\dot q^1 + \omega^2 \wedge d\dot q^2 + \omega^3 \wedge d\dot q^3) + F, \tag{48}$$

where F is any 2-contact 2-form. Corresponding dynamical form is

$$E = mG\,dq^3 \wedge dt + \sum_{\sigma=1}^{3} m\ddot q^\sigma\, dq^\sigma \wedge dt. \tag{49}$$

The restriction of motion is given by equation

$$f \equiv (\dot q^1)^2 + (\dot q^2)^2 + (\dot q^3)^2 - C = 0, \tag{50}$$

which defines the constraint submanifold Q in J^1Y. Let $U \subset J^1Y$ be the set of all points where $\dot q^3 > 0$ and consider on U the adapted coordinates $(t, q^1, q^2, q^3, \dot q^1, \dot q^2, \bar f)$, where $\bar f = \dot q^3 - g$, $g = \sqrt{C - (\dot q^1)^2 - (\dot q^2)^2}$ is equation of the constraint (50) in normal form.

The constrained system $[\alpha_Q]$ related to the mechanical system $[\alpha]$ and the constraint Q is the equiavalence class of the 2-forms

$$\alpha_Q = \sum_{l=1,2} A'_l\, \omega^l \wedge dt + \sum_{l,s=1,2} B'_{ls}\, \omega^l \wedge d\dot q^s + F + \varphi_{(2)} \tag{51}$$

on Q, where

$$A'_l = \left[m\,G\,\frac{\dot q^l}{\dot q^3} \right]_\iota = m\,G\,\frac{\dot q^l}{g} \qquad 1 \leq l \leq 2,$$

$$B'_{ls} = \left[-\, m\left(\delta_{ls} + \frac{\dot q^l \dot q^s}{(\dot q^3)^2} \right) \right]_\iota \; = \; -\, m\left(\delta_{ls} + \frac{\dot q^l \dot q^s}{g^2} \right) \qquad 1 \leq l, s \leq 2,$$

and F is any 2-contact 2-form and $\varphi_{(2)}$ is any constraint 2-form. The constrained system $[\alpha_Q]$ is regular since the matrix (B'_{ls}) is identical as in the Example 3.Motion of this constrained system is described by two equations

$$\begin{aligned} &\left[m\,G\,\frac{\dot q^1}{g} + m\left(1 + \frac{(\dot q^1)^2}{g^2} \right)\ddot q^1 + m\frac{\dot q^1 \dot q^2}{g^2}\,\ddot q^2 \right] \circ J^2\bar\gamma = 0, \\ &\left[m\,G\,\frac{\dot q^2}{g} + m\left(1 + \frac{(\dot q^2)^2}{g^2} \right)\ddot q^2 + m\frac{\dot q^1 \dot q^2}{g^2}\,\ddot q^1 \right] \circ J^2\bar\gamma = 0, \end{aligned} \tag{52}$$

for the Q-admissible sections $\bar\gamma = (t, q^1(t), q^2(t), q^3(t))$ satisfying the constraint equation

$$\dot q^3 = \sqrt{C - (\dot q^1)^2 - (\dot q^2)^2}\,.$$

After simple arrangements equations of motion of the constrained system become

$$\begin{aligned} \ddot q^1(t) &= \frac{G}{C}\,\dot q^1\,\sqrt{C - (\dot q^1)^2 - (\dot q^2)^2}\,, \\ \ddot q^2(t) &= \frac{G}{C}\,\dot q^2\,\sqrt{C - (\dot q^1)^2 - (\dot q^2)^2}\,, \\ \dot q^3 &= \sqrt{C - (\dot q^1)^2 - (\dot q^2)^2}\,, \end{aligned} \tag{53}$$

The same equations are derived in [16] by different method. The above system of differential equations can be reduced on first order system

$$
\begin{aligned}
&\dot{p}^1(t) = D\, p^1 \sqrt{C-(p^1)^2-(p^2)^2}\,,\\
&\dot{p}^2(t) = D\, p^2 \sqrt{C-(p^1)^2-(p^2)^2}\,,\\
&\dot{q}^3 = \sqrt{C-(p^1)^2-(p^2)^2}\,,
\end{aligned}
\tag{54}
$$

where $D = G/C$. If p^2 is not zero, then $p^1/p^2 = \kappa$ is first integral and consequently we can separate equations for p^1 and p^2 and integrate

$$
\int \frac{dp^1}{p^1 \sqrt{C-\left(1+\dfrac{1}{\kappa^2}\right)(p^1)^2}} = \int D\, dt
$$

$$
\int \frac{dp^2}{p^2 \sqrt{C-(1+\kappa^2)(p^2)^2}} = \int D\, dt.
$$

After integration and isolation variables p^1, p^2 we obtain

$$
p^1 = \frac{dq^1}{dt} = \kappa \sqrt{\frac{C}{1+\kappa^2}}\, \frac{1}{\sqrt{(B_1 e^{(G/\sqrt{C})t}-1)^2+1}}
$$

$$
p^2 = \frac{dq^2}{dt} = \sqrt{\frac{C}{1+\kappa^2}}\, \frac{1}{\sqrt{(B_2 e^{(G/\sqrt{C})t}-1)^2+1}}\,,
$$

where integrating constants $B_1 = B_2$, since $p^1/p^2 = \kappa$. Now we find the primitive function

$$
\int \frac{dt}{\sqrt{(B_1e^{\alpha t}-1)^2+1}} = \frac{\sqrt{2}}{4\alpha} \ln\left[\frac{\sqrt{(B_1e^{\alpha t}-1)^2+1}-(B_1e^{\alpha t}-1)-(1+\sqrt{2})}{\sqrt{(B_1e^{\alpha t}-1)^2+1}-(B_1e^{\alpha t}-1)-(1-\sqrt{2})}\right],
$$

where $\alpha = G/\sqrt{C}$. Finally after reparametrizing $\tau = B_1 e^{\alpha t} - 1$ we obtain solution

$$
q^1(t) = \frac{\kappa}{4\alpha}\sqrt{\frac{2C}{1+\kappa^2}}\, \ln\left[\frac{\sqrt{\tau^2+1}-\tau-(1+\sqrt{2})}{\sqrt{\tau^2+1}-\tau-(1-\sqrt{2})}\right] + A_1
$$

$$
q^2(t) = \frac{1}{4\alpha}\sqrt{\frac{2C}{1+\kappa^2}}\, \ln\left[\frac{\sqrt{\tau^2+1}-\tau-(1+\sqrt{2})}{\sqrt{\tau^2+1}-\tau-(1-\sqrt{2})}\right] + A_2
$$

$$
q^3(t) = \frac{\sqrt{2C}}{2\alpha} \ln\left[\frac{(\sqrt{2}-1)+\sqrt{\tau^2+1}-\tau}{(\sqrt{2}+1)-\sqrt{\tau^2+1}+\tau}\right] - \frac{\sqrt{C}}{\alpha}\ln\left[\sqrt{\tau^2+1}-\tau\right] + A_3
$$

Example 4. (see [16], pp.992, Example 4.3.) *Consider a particle of the mass m in homogenous gravitational field (the same as in previous Example). Movement of a particle is now subjected to nonholonomic condition* $b^2((\dot{q}^1)^2+(\dot{q}^2)^2)-(\dot{q}^3)^2=0$, where b is a constant.

This mechanical system is quite identical as in Example 3, i.e. it is represented by the Lepage form (48). However the constraint condition

$$
f \equiv b^2((\dot{q}^1)^2+(\dot{q}^2)^2)-(\dot{q}^3)^2 = 0. \tag{55}
$$

or equivalently in normal form

$$\dot q^3 = g = b\,\sqrt{(\dot q^1)^2 + (\dot q^2)^2} \tag{56}$$

is different.

Corresponding constrained mechanical system is given by equivalence class $[\alpha_Q]$ of 2-forms (51), where

$$A'_l = \left[-m\,G\,\frac{b^2\dot q^l}{\dot q^3}\right]_\iota = -m\,G\,\frac{b\,\dot q^l}{\sqrt{(\dot q^1)^2+(\dot q^2)^2}} \quad 1\le l\le 2,$$

$$B'_{ls} = \left[-m\left(\delta_{ls}+b^4\frac{\dot q^l\dot q^s}{(\dot q^3)^2}\right)\right]_\iota = -m\left(\delta_{ls}+b^2\frac{\dot q^l\dot q^s}{(\dot q^1)^2+(\dot q^2)^2}\right) \quad 1\le l,s\le 2.$$

The reduced equations of motion are the following system of ODE's of the second order

$$\left[m\,G\frac{b\,\dot q^1}{\sqrt{(\dot q^1)^2+(\dot q^2)^2}}+m\left(1+b^2\frac{(\dot q^1)^2}{(\dot q^1)^2+(\dot q^2)^2}\right)\ddot q^1+mb^2\frac{\dot q^1\dot q^2}{(\dot q^1)^2+(\dot q^2)^2}\ddot q^2\right]\circ J^2\bar\gamma=0,$$

$$\left[m\,G\frac{b\,\dot q^2}{\sqrt{(\dot q^1)^2+(\dot q^2)^2}}+m\left(1+b^2\frac{(\dot q^2)^2}{(\dot q^1)^2+(\dot q^2)^2}\right)\ddot q^2+mb^2\frac{\dot q^1\dot q^2}{(\dot q^1)^2+(\dot q^2)^2}\ddot q^1\right]\circ J^2\bar\gamma=0,$$

for the Q-admissible sections $\bar\gamma = (t, q^1(t), q^2(t), q^3(t))$ satisfying the constraint equation (56). Isolating the second derivatives we obtain

$$\begin{aligned}\ddot q^1(t) &= -\,b\,G\,\frac{\dot q^1}{(1+b^2)\sqrt{(\dot q^1)^2+(\dot q^2)^2}},\\ \ddot q^2(t) &= -\,b\,G\,\frac{\dot q^2}{(1+b^2)\sqrt{(\dot q^1)^2+(\dot q^2)^2}}.\end{aligned} \tag{57}$$

The same equations are derived in [16] by different method.

Now we proceed to the method of solution of these equations. First we diferentiate the constraint equation (56)

$$\ddot q^3 = \frac{b^2}{\dot q^3}\,(\dot q^1\,\ddot q^1 + \dot q^2\,\ddot q^2)$$

and substituting the reduced equations (57) we obtain the equality

$$\ddot q^3 = -\,\frac{G\,b^2}{1+b^2},$$

which can be simple integrated

$$\dot q^3 \equiv b\,\sqrt{(\dot q^1)^2+(\dot q^2)^2} = -\,\frac{G\,b^2}{1+b^2}\,t + K^3_1.$$

Finally we substitute the last equality back to (57) and we obtain simple differential equations, which can be reduced on the first oder equations with separable variables. Solution of this problem we get in the form

$$q^1 = -\,\frac{1}{2}\,\frac{G\,b^2}{1+b^2}\,K^1_1\,t^2 + K^1_1 K^3_1\,t + K^1_2,$$

$$q^2 = -\frac{1}{2}\,\frac{G\,b^2}{1+b^2}\,K_1^2\,t^2 + K_1^2 K_1^3\,t + K_2^2,$$

$$q^3 = -\frac{1}{2}\,\frac{G\,b^2}{1+b^2}\,t^2 + K_1^3\,t + K_2^3,$$

where K_j^i are constants and identity $(K_1^1)^2 + (K_1^2)^2 = 1/b^2$ holds.

Example 5. *Consider a particle of the mass m moving in central gravitational field such a way, that square of the magnitude of the instantaneous velocity is constant at each moment.*

We consider $Y = \mathbb{R}\times\mathbb{R}^3$, the standard configuration space for the mooving particle in threedimensional space and $(t, q^\sigma), 1 \leq \sigma \leq 3$, fiber coordinates on it. Potential energy of attractive gravitational field is $V = -a/\rho$, where a is a constant of gravitational interaction and $\rho = \sqrt{(q^1)^2 + (q^2)^2 + (q^3)^2}$ is distance of the particle from the point center of gravitational field. Lagrange function in this example has the form

$$L = \frac{1}{2}\,m((\dot q^1)^2 + (\dot q^1)^2 + (\dot q^3)^2) + \frac{a}{\rho}. \tag{58}$$

Dynamical form corresponding to the Lagrangian $\lambda = L\,dt$ is then expressed by

$$E = \sum_{\sigma=1}^{3}\left(m\ddot q^\sigma + \frac{a\,q^\sigma}{\rho^3}\right) dq^\sigma \wedge dt. \tag{59}$$

Considered mechanical system related with the dynamical form (59) is represented by equivalence class $[\alpha]$ of 2- forms on J^1Y

$$\alpha = -\sum_{\sigma=1}^{3}\frac{a\,q^\sigma}{\rho^3}\,\omega^\sigma \wedge dt - m\sum_{\sigma=1}^{3}\omega^\sigma \wedge d\dot q^\sigma + F. \tag{60}$$

However motion of this mechanical system is restricted by one nonholonomic constraint given by equation (50), which can be expressed locally in explicit form

$$\dot q^3 = \sqrt{C - (\dot q^1)^2 - (\dot q^2)^2} \tag{61}$$

quite identically as in Example 3.

Corresponding constrained mechanical system is given by equivalence class $[\alpha_Q]$ of 2-forms (51), where

$$A'_l = -\frac{a\,q^l}{\rho^3} + \frac{a\,q^3}{\rho^3}\,\frac{\dot q^l}{\sqrt{C - (\dot q^1)^2 - (\dot q^2)^2}} \qquad 1 \leq l \leq 2,$$

$$B'_{11} = -\frac{m\,(C - (\dot q^2)^2)}{C - (\dot q^1)^2 - (\dot q^2)^2}, \qquad B'_{22} = -\frac{m\,(C - (\dot q^1)^2)}{C - (\dot q^1)^2 - (\dot q^2)^2},$$

$$B'_{12} = B'_{21} = -\frac{m\,\dot q^1\,\dot q^2}{C - (\dot q^1)^2 - (\dot q^2)^2}.$$

Motion of this constrained system is described by two equations

$$\left[-\frac{a\,q^1}{\rho^3} + \frac{a\,q^3}{\rho^3}\,\frac{\dot q^1}{g} - m\left(1 + \frac{(\dot q^1)^2}{g^2}\right)\ddot q^1 - \frac{m\,\dot q^1\,\dot q^2}{g^2}\,\ddot q^2\right] \circ J^2\bar\gamma = 0,$$

$$\left[-\frac{a\,q^2}{\rho^3}+\frac{a\,q^3}{\rho^3}\,\frac{\dot q^2}{g}-m\left(1+\frac{(\dot q^2)^2}{g^2}\right)\ddot q^2-\frac{m\,\dot q^1\,\dot q^2}{g^2}\,\ddot q^1\right]\circ J^2\bar\gamma=0,$$

for the Q-admissible sections $\bar\gamma = (t, q^1(t), q^2(t), q^3(t))$ satisfying the constraint equation (61).

Example 6. (see [21], pp. 55) *Consider a disc rolling without sliding on the horizontal plane. Let $Oxyz$ be a fixed orthogonal system of coordinates with x and y-axis in the horizontal plane and z-axis directed vertically upward. Then the position of the disc on the plane may be specified by the five generalized coordinates $x, y, \psi, \phi, \vartheta$, where x and y are the coordinates of the point P of contact of the disc and the horizontal plane, ψ is the angle of proper rotation of the disc, ϕ is the angle between the the tangent to the disc at the point P and the x-axis, and ϑ is the angle between the rotating axis of the disk and the parallel line to z-axis which is going through the point P (i.e. $\pi/2-\vartheta$ is the angle of inclination between the plane of the disc and horizontal plane).*

So the base space $X = \mathbb{R}$, the configuration space is $Y = \mathbb{R}\times\mathbb{R}^2\times S^1\times S^1\times S^1$ and and phase space is $J^1Y = \mathbb{R}\times\mathbb{R}^2\times S^1\times S^1\times S^1\times\mathbb{R}^2\times S^1\times S^1\times S^1$. Hence the fiber coordinates on Y are denoted by $(t, x, y, \psi, \phi, \vartheta)$ and the associated coordinates on J^1Y are denoted by $(t, x, y, \psi, \phi, \vartheta, \dot x, \dot y, \dot\psi, \dot\phi, \dot\vartheta)$.

The Lagrange function of this mechanical system is given by relation $L = T - V$. The kinetic energie T is given by the sum of energy of sliding and rotating motion of the disk:

$$\begin{aligned}
T = {}&\tfrac{1}{2}\,m(\dot x^2+\dot y^2+R^2\dot\vartheta^2+R^2\dot\phi^2\sin^2\vartheta)-\\
&-mR(\dot\vartheta\cos\vartheta\,(\dot x\sin\phi-\dot y\cos\phi)+\dot\phi\sin\vartheta\,(\dot x\cos\phi+\dot y\sin\phi))+\\
&+\tfrac{1}{2}\,I_1(\dot\vartheta^2+\dot\phi^2\cos^2\vartheta)+\tfrac{1}{2}\,I_2(\dot\psi+\dot\phi\sin\vartheta)^2,
\end{aligned}\tag{62}$$

where m is the mass, and I_1, I_2 are the principal moments of inertia of the disk. The potential energy of the disk is $V = mgR\cos\vartheta$.

If we compute the motion equation (5) of this Lagrangian system according to (9) and (10), where $1 \le \sigma, \rho \le 5$ and coordinates $(q^1, q^2, q^3, q^4, q^5)$ are substituted by corresponding cordinates $(x, y, \psi, \phi, \vartheta)$ we obtain the following five Euler- Lagrange equations:

$$\begin{aligned}
&m\ddot x-mR((\cos\phi\sin\vartheta)\,\ddot\phi+(\sin\phi\cos\vartheta)\,\ddot\vartheta)+\\
&\quad+mR((\sin\phi\sin\vartheta)\,(\dot\phi^2+\dot\vartheta^2)-(2\cos\phi\cos\vartheta)\,\dot\phi\,\dot\vartheta)=0,\\
&m\ddot y-mR((\sin\phi\sin\vartheta)\,\ddot\phi-(\cos\phi\cos\vartheta)\,\ddot\vartheta)-\\
&\quad-mR((\cos\phi\sin\vartheta)\,(\dot\phi^2+\dot\vartheta^2)+(2\sin\phi\cos\vartheta)\,\dot\phi\,\dot\vartheta)=0,\\
&I_2\,(\ddot\psi+\sin\vartheta\,\ddot\phi)+(I_2\cos\vartheta)\,\dot\phi\,\dot\vartheta=0,\\
&mR((\cos\phi\sin\vartheta)\,\ddot x+(\sin\phi\sin\vartheta)\,\ddot y)-(I_2\sin\vartheta)\,\ddot\psi-\\
&\quad-((mR^2+I_2)\sin^2\vartheta+I_1\cos^2\vartheta)\ddot\phi-\\
&\quad-(I_2\cos\vartheta)\,\dot\psi\,\dot\vartheta-2(mR^2-I_1+I_2)\,(\sin\vartheta\cos\vartheta)\,\dot\phi\,\dot\vartheta=0,\\
&mR((\sin\phi\cos\vartheta)\,\ddot x-(\cos\phi\cos\vartheta)\,\ddot y)-(mR^2+I_1)\,\ddot\vartheta+\\
&\quad+(mR^2-I_1+I_2)\,(\sin\vartheta\cos\vartheta)\,\dot\phi^2+(I_2\cos\vartheta)\,\dot\psi\,\dot\phi+mgR\sin\vartheta=0.
\end{aligned}\tag{63}$$

The condition that the disk rolls without sliding on the horizontal plane means that the instantaneous velocity of the point of contact of the disk is equal to zero at all times. This gives rise to the following nonholonomic constraints

$$f^1\equiv\dot x-R\cos\phi\,\dot\psi=0,\qquad f^2\equiv\dot y-R\sin\phi\,\dot\psi=0,\tag{64}$$

or in normal form

$$\dot{x} = g^1 \equiv R\cos\phi\,\dot{\psi}, \quad \dot{y} = g^2 \equiv R\sin\phi\,\dot{\psi}.$$

Equations (64) defined constraint submanifold $Q \subset J^1Y$, since the condition (14) is satisfied, i.e.

$$\operatorname{rank}\left(\frac{\partial f^i}{\partial \dot{q}^\sigma}\right) = \operatorname{rank}\begin{pmatrix} 1 & 0 & -R\cos\phi & 0 & 0 \\ 0 & 1 & -R\sin\phi & 0 & 0 \end{pmatrix} = 2.$$

Thus $\dim Q = \dim J^1Y - 2 = 9$. The constraint 1-forms (16) are in this case the following two forms

$$\varphi^1 = dx - R\cos\phi\, d\psi, \quad \varphi^2 = dy - R\sin\phi\, d\psi. \tag{65}$$

Now one can construct the constrained system $[\alpha_Q]$ related to the mechanical system $[\alpha]$ and the constraint Q as the equiavalence class of the 2-form

$$\begin{aligned} \alpha_Q = {} & A'_1\,\omega^1 \wedge dt + A'_2\,\omega^2 \wedge dt + A'_3\,\omega^3 \wedge dt + \\ & + \sum_{l=1}^{3} B'_{l1}\,\omega^l \wedge d\dot{\psi} + B'_{l2}\,\omega^l \wedge d\dot{\phi} + B'_{l3}\,\omega^l \wedge d\dot{\vartheta} + F + \varphi_{(2)} \end{aligned} \tag{66}$$

on Q, where $\omega^1 = d\psi - \dot{\psi}dt$, $\omega^2 = d\phi - \dot{\phi}dt$, $\omega^3 = d\vartheta - \dot{\vartheta}dt$ and where F is a 2-contact 2-form and $\varphi_{(2)}$ is a constraint 2-form. Computing the coefficients A'_l according to (27) we obtain following expressions:

$$\begin{aligned} A'_1 &= (2mR^2 - I_2)(\cos\vartheta)\,\dot{\phi}\,\dot{\vartheta}, \\ A'_2 &= -I_2\cos\vartheta\,\dot{\psi}\,\dot{\vartheta} - 2(mR^2 - I_1 + I_2)(\sin\vartheta\cos\vartheta)\,\dot{\phi}\,\dot{\vartheta}, \\ A'_3 &= (I_2 - mR^2)\cos\vartheta\,\dot{\psi}\,\dot{\phi} + (mR^2 - I_1 + I_2)(\sin\vartheta\cos\vartheta)\,\dot{\phi}^2 + mgR\sin\vartheta, \end{aligned}$$

and coefficients B'_{ls} according to (28) are

$$\begin{aligned} B'_{11} &= -(mR^2 + I_2), & B'_{12} &= B'_{21} = (mR^2 - I_2)\sin\vartheta, \\ B'_{22} &= -(mR^2 + I_2)\sin^2\vartheta - I_1\cos^2\vartheta, & B'_{23} &= B'_{32} = 0, \\ B'_{33} &= -(mR^2 + I_1), & B'_{31} &= B'_{13} = 0. \end{aligned}$$

Hence the reduced equations of motion (23) of the constrained system $[\alpha_Q]$ take the form:

$$\begin{aligned} & (mR^2 + I_2)\,\ddot{\psi} + (I_2 - mR^2)(\sin\vartheta)\ddot{\phi} + (I_2 - 2mR^2)(\cos\vartheta)\dot{\phi}\,\dot{\vartheta} = 0, \\ & (mR^2 - I_2)(\sin\vartheta)\ddot{\psi} - ((mR^2 + I_2)\sin^2\vartheta + I_1\cos^2\vartheta)\ddot{\phi} - \\ & \quad - I_2(\cos\vartheta)\dot{\psi}\,\dot{\vartheta} - 2(mR^2 - I_1 + I_2)(\sin\vartheta\cos\vartheta)\dot{\phi}\,\dot{\vartheta} = 0, \\ & (mR^2 + I_1)\ddot{\vartheta} - (mR^2 - I_1 + I_2)(\sin\vartheta\cos\vartheta)\dot{\phi}^2 - \\ & \quad - (I_2 - mR^2)(\cos\vartheta)\dot{\psi}\dot{\phi} - mgR\sin\vartheta = 0. \end{aligned} \tag{67}$$

Example 7. (see [21], pp. 131, Example 3) *Consider a homogeneous sphere of radius R rolling without sliding on a horizontal plane which rotates with non constant angular velocity $\Omega(t)$ about a vertical axis. We assume that except the constant gravitional force, no other external forces to act on the sphere.*

Let the z-axis of the fixed system of coordinates $Oxyz$ coincide with the axis of rotation. Let (x, y) denote the position of contact of the sphere with the plane and ϑ, φ, ψ denote Euler angles of rotating sphere. The angle ϑ is the angle of inclination, the φ

is rotating angle and ψ is the angle of precession. Hence $(t, x, y, \vartheta, \varphi, \psi)$ are the fiber coordinates on configuration space $Y = \mathbb{R} \times \mathbb{R}^2 \times SO(3)$, where $SO(3)$ is special orthogonal group parametrized by Euler angles and $(t, x, y, \vartheta, \varphi, \psi, \dot{x}, \dot{y}, \dot{\vartheta}, \dot{\varphi}, \dot{\psi})$ are associated coordinates on $J^1Y = \mathbb{R} \times \mathbb{R}^2 \times SO(2) \times \mathbb{R}^2 \times SO(2)$.

The potencial energy is constant, so without loss of generality we put $V = 0$. In addition, since we do not consider external forces, the Lagrange function is given by the kinetic energy of the rotating sphere

$$L = T = \tfrac{1}{2}\,(\dot{x}^2 + \dot{y}^2 + k^2(\dot{\vartheta}^2 + \dot{\varphi}^2 + \dot{\psi}^2 + 2\dot{\varphi}\,\dot{\psi}\cos\vartheta)), \tag{68}$$

where k is the radius of gyration and the mass of the sphere is $m = 1$.

The motion equations (5) of this Lagrangian system in coordinates $(q^1, q^2, q^3, q^4, q^5) = (x, y, \vartheta, \varphi, \psi)$ become:

$$\begin{aligned}
&\ddot{x} = 0,\\
&\ddot{y} = 0,\\
&k^2(\ddot{\vartheta} + \sin\vartheta\,\dot{\varphi}\,\dot{\psi}) = 0,\\
&k^2(\ddot{\varphi} + \cos\vartheta\,\ddot{\psi} - \sin\vartheta\,\dot{\vartheta}\,\dot{\psi}) = 0,\\
&k^2(\cos\vartheta\,\ddot{\varphi} + \ddot{\psi} - \sin\vartheta\,\dot{\vartheta}\,\dot{\varphi}) = 0.
\end{aligned} \tag{69}$$

Denoting by ω the instantaneous angular velocity of the sphere, we write down the condition of rolling without sliding of the sphere on the rotating plane

$$\dot{x} - R\omega_y + \Omega(t)\,y = 0, \qquad \dot{y} + R\omega_x - \Omega(t)\,x = 0, \tag{70}$$

or, using the Euler angles we obtain following two equations

$$\begin{aligned}
&f^1 \equiv \dot{x} - R\sin\psi\,\dot{\vartheta} + R\sin\vartheta\cos\psi\,\dot{\varphi} + \Omega(t)\,y = 0,\\
&f^2 \equiv \dot{y} + R\cos\psi\,\dot{\vartheta} + R\sin\vartheta\sin\psi\,\dot{\varphi} - \Omega(t)\,x = 0,
\end{aligned} \tag{71}$$

which represent two nonholonomic constraints. These equations evidently satisfy the condition (14)

$$\operatorname{rank}\left(\frac{\partial f^i}{\partial \dot{q}^\sigma}\right) = \operatorname{rank}\begin{pmatrix} 1 & 0 & -R\sin\psi & R\sin\vartheta\cos\psi & 0\\ 0 & 1 & -R\sin\varphi & R\sin\vartheta\cos\psi & 0 \end{pmatrix} = 2,$$

thus $\dim Q = \dim J^1Y - 2 = 9$. The constraint 1-forms (16) take the form

$$\begin{aligned}
&\varphi^1 = dx + \Omega(t)\,y\,dt - R\sin\psi\,d\vartheta + R\sin\vartheta\cos\psi\,d\varphi,\\
&\varphi^2 = dy - \Omega(t)\,x\,dt + R\cos\psi\,d\vartheta + R\sin\vartheta\sin\psi\,d\varphi.
\end{aligned}$$

The constrained system $[\alpha_Q]$ is in this case represented by the equiavalence class of the 2-form

$$\begin{aligned}
\alpha_Q = {} & A'_1\,\omega^1 \wedge dt + A'_2\,\omega^2 \wedge dt + A'_3\,\omega^3 \wedge dt + \\
& + \sum_{l=1}^{3} B'_{l1}\,\omega^l \wedge d\dot{\vartheta} + B'_{l2}\,\omega^l \wedge d\dot{\varphi} + B'_{l3}\,\omega^l \wedge d\dot{\psi} + F + \varphi_{(2)}
\end{aligned} \tag{72}$$

on Q, where $\omega^1 = d\vartheta - \dot{\vartheta}\,dt$, $\omega^2 = d\varphi - \dot{\varphi}\,dt$, $\omega^3 = d\psi - \dot{\psi}\,dt$ and where for coefficients

A'_l we obtain

$$\begin{aligned}
A'_1 =& -(R^2+k^2)\,\dot\varphi\,\dot\psi\,\sin\vartheta + \\
&+ R\Omega(t)(\dot x\cos\psi + \dot y\sin\psi) + R\dot\Omega(t)(x\cos\psi + y\sin\psi),\\
A'_2 =& -R^2\dot\vartheta\,\dot\varphi\sin\vartheta\cos\vartheta + (R^2+k^2)\,\dot\vartheta\,\dot\psi\,\sin\vartheta + \\
&+ R\dot\Omega(t)\sin\vartheta\,(x\sin\psi - y\cos\psi) + R\Omega(t)\sin\vartheta\,(\dot x\sin\psi - \dot y\cos\psi),\\
A'_3 =& k^2\dot\vartheta\,\dot\varphi\sin\vartheta,
\end{aligned}$$

and for coefficients B'_{ls} we have

$$\begin{array}{lll}
B'_{11} = -(R^2+k^2), & B'_{12} = 0, & B'_{13} = 0,\\
B'_{21} = 0, & B'_{22} = -(R^2\sin^2\vartheta + k^2), & B'_{23} = -k^2\cos\vartheta\\
B'_{31} = 0, & B'_{32} = -k^2\cos\vartheta, & B'_{33} = -k^2.
\end{array}$$

The motion of this constrained system is described by this three equations

$$\begin{aligned}
&(R^2+k^2)\,\ddot\vartheta + (R^2+k^2)\,\dot\varphi\,\dot\psi\sin\vartheta - \\
&\quad - R\,\Omega(t)(\dot x\cos\psi + \dot y\sin\psi) - R\,\dot\Omega(t)(x\cos\psi + y\sin\psi) = 0,\\
&(R^2\sin^2\vartheta + k^2)\,\ddot\varphi + k^2\cos\vartheta\,\ddot\psi + R^2\dot\vartheta\,\dot\varphi\sin\vartheta\cos\vartheta - (R^2+k^2)\,\dot\vartheta\,\dot\psi\sin\vartheta\\
&\quad - R\,\Omega(t)\sin\vartheta\,(\dot x\sin\psi - \dot y\cos\psi) - R\,\dot\Omega(t)\sin\vartheta\,(x\sin\psi - y\cos\psi) = 0,\\
&k^2\cos\vartheta\ddot\phi + k^2\ddot\psi - k^2\dot\vartheta\,\dot\varphi\sin\vartheta = 0.
\end{aligned}\tag{73}$$

To simlify these equations we can use another coordinates, so called *quasi-coordinates.* Recall that $\omega_x, \omega_y, \omega_z$ denote the components of the instantaneous angular velocity, which are determined by means of the Euler's angles

$$\begin{aligned}
\omega_x &= \dot\vartheta\cos\psi + \dot\varphi\sin\vartheta\sin\psi,\\
\omega_y &= \dot\vartheta\sin\psi - \dot\varphi\sin\vartheta\cos\psi,\\
\omega_z &= \dot\psi + \dot\varphi\cos\vartheta.
\end{aligned}\tag{74}$$

Consider now q^1, q^2, q^3 quasi-coordinates on the configuration space, such that $\dot q^1 = \omega_x, \dot q^2 = \omega_y, \dot q^3 = \omega_z$. Denote by $(t, x, y, q^1, q^2, q^3, \dot x, \dot y, \omega_x, \omega_y, \omega_z)$ associated coordinates on J^1Y. Then the expression of the Lagrangian (68) by means of quasi-coordinates is following

$$L = \tfrac{1}{2}\,(\dot x^2 + \dot y^2 + k^2(\omega_x^2 + \omega_y^2 + \omega_z^2)),$$

the equations of constrained submanifold are of the form (70) and the constraint 1-forms take the form

$$\phi^1 = dx - R\,dq^2, \quad \phi^2 = dy + R\,dq^1.$$

The reduced equations of motion of the constrained mechanical system in quasi-coordinates take the form

$$\begin{aligned}
&(R^2+k^2)\ddot q^1 - R^2\Omega(t)\,\dot q^2 - R\dot\Omega(t)\,x + R\Omega^2(t)\,y = 0,\\
&(R^2+k^2)\ddot q^2 + R^2\Omega(t)\,\dot q^1 - R\dot\Omega(t)\,y - R\Omega^2(t)\,x = 0,\\
&k\,\ddot q^3 = 0.
\end{aligned}\tag{75}$$

Using the definition of the quasi-coordinates q^1, q^2, q^3 we obtain that

$$\dot{q}^3 = \omega_z = C_3 = const.$$

and first two equations of the system (75) we can reduce to system of linear diferential equations of the first order

$$(76)\qquad \begin{aligned} &(R^2+k^2)\dot{\omega}_x - R^2\Omega(t)\,\omega_y - R\,\dot{\Omega}(t)\,x + R\,\Omega^2(t)\,y = 0,\\ &(R^2+k^2)\dot{\omega}_y + R^2\Omega(t)\,\omega_x - R\,\dot{\Omega}(t)\,y - R\,\Omega^2(t)\,x = 0, \end{aligned}$$

When we substitute the constraint equations (70) into equations (75) we get two first integrals:

$$(77)\qquad \begin{aligned} &(R^2+k^2)\omega_x - R\,\Omega(t)\,x = D_1(R^2+k^2),\\ &(R^2+k^2)\omega_y - R\,\Omega(t)\,y = D_2(R^2+k^2), \end{aligned}$$

where D_1, D_2 are arbitrary constants. Comparing the expressions for ω_x, ω_y from the constraint equations (70) and from (77) we obtain

$$(78)\qquad \dot{x} + \frac{k^2\Omega(t)}{R^2+k^2}\,y + R\,D_1 = 0, \qquad \dot{y} - \frac{k^2\Omega(t)}{R^2+k^2}\,x - R\,D_2 = 0$$

and diferentiating we get the following system of diferential equations of the second order

$$(79)\qquad \ddot{x} + \frac{k^2\Omega(t)}{R^2+k^2}\,\dot{y} + \frac{k^2\dot{\Omega}(t)}{R^2+k^2}\,y = 0, \qquad \ddot{y} - \frac{k^2\Omega(t)}{R^2+k^2}\,\dot{x} + \frac{k^2\dot{\Omega}(t)}{R^2+k^2}\,x = 0$$

for unknown functions $x(t), y(t)$, which describe the motion of the point of contact of the sphere with the plane.

Let us suppose, that for given function $\Omega(t)$ of the angular velocity of the rotating of the plane we found a solution $x(t), y(t)$ of the system (79). Put

$$A = (R^2+k^2), \qquad b(t) = R^2\,\Omega(t),$$

$$F_1(t, x(t), y(t)) = R\,\dot{\Omega}(t)\,x - R\,\Omega^2(t)\,y,$$

$$F_2(t, x(t), y(t)) = R\,\dot{\Omega}(t)\,y + R\,\Omega^2(t)\,x,$$

then the system (76) can be written in the form:

$$(80)\qquad \begin{aligned} &A\,\dot{\omega}_x - b(t)\,\omega_y = F_1(t, x(t), y(t))\\ &A\,\dot{\omega}_y + b(t)\,\omega_x = F_2(t, x(t), y(t)). \end{aligned}$$

Evidently, it represents system of two linear non-homogenous dirərential equations of the first order with non-constant coefficients. First one solves corresponding homogenous system

$$\dot{\omega}_x = \frac{B(t)}{A}\,\omega_y \qquad \dot{\omega}_y = -\frac{B(t)}{A}\,\omega_x.$$

We obtain following result:

$$(81)\qquad \begin{aligned} &\omega_x^H(t) = C_1 \sin\left(\frac{B(t)}{A}\right) + C_2 \cos\left(\frac{B(t)}{A}\right),\\ &\omega_y^H(t) = -C_2 \sin\left(\frac{B(t)}{A}\right) + C_1 \cos\left(\frac{B(t)}{A}\right), \end{aligned}$$

where $B(t)=\int b(t)\,dt$. Particular solution we are looking for by the standart procedure of variation of the constants

$$(82)\qquad \begin{aligned} \omega_x^P(t) &= C_1(t)\sin\left(\frac{B(t)}{A}\right)+C_2(t)\cos\left(\frac{B(t)}{A}\right),\\ \omega_y^P(t) &= C_1(t)\cos\left(\frac{B(t)}{A}\right)-C_2(t)\sin\left(\frac{B(t)}{A}\right), \end{aligned}$$

where $C_1(t),C_2(t)$ we get integrating of the equations

$$(83)\qquad \begin{aligned} \dot C_1(t) &= F_1(t,x(t),y(t))\sin\left(\frac{B(t)}{A}\right)+F_2(t,x(t),y(t))\cos\left(\frac{B(t)}{A}\right),\\ \dot C_2(t) &= F_1(t,x(t),y(t))\cos\left(\frac{B(t)}{A}\right)-F_2(t,x(t),y(t))\sin\left(\frac{B(t)}{A}\right). \end{aligned}$$

General solution of the system (80) is then of the form

$$\begin{pmatrix}\omega_x(t)\\ \omega_y(t)\end{pmatrix}=\begin{pmatrix}\omega_x^H(t)\\ \omega_y^H(t)\end{pmatrix}+\begin{pmatrix}\omega_x^P(t)\\ \omega_y^P(t)\end{pmatrix}.$$

The solution of this example by means of quasi-coordinates is then determined by the elementary quadratures

$$q^1(t)=\int\omega_x(t)\,dt,\qquad q^2(t)=\int\omega_y(t)\,dt,\qquad q^3(t)=\int C_3\,dt,$$

and the solution by means of Euler's angles is desribed by the system of diferential equations (74).

In particular case, when $\Omega(t)=\Omega_0=const.$, (see [21]) the system (79) becomes

$$(84)\qquad \ddot x+\frac{k^2\Omega_0}{R^2+k^2}\,\dot y=0,\qquad \ddot y-\frac{k^2\Omega_0}{R^2+k^2}\,\dot x=0.$$

Using first integrals (78) we write:

$$(85)\qquad \begin{aligned} \ddot x+\left(\frac{k^2\Omega_0}{R^2+k^2}\right)^2 x &= -\,\frac{k^2R\,\Omega_0}{R^2+k^2}\,D_2,\\ \ddot y+\left(\frac{k^2\Omega_0}{R^2+k^2}\right)^2 y &= -\,\frac{k^2R\,\Omega_0}{R^2+k^2}\,D_1. \end{aligned}$$

A solution of the corresponding homogeneous system is:

$$x^H(t)=A_1\sin\left[\left(\frac{k^2\Omega_0}{R^2+k^2}\right)^2 t\right]+A_2\cos\left[\left(\frac{k^2\Omega_0}{R^2+k^2}\right)^2 t\right],$$

$$y^H(t)=A_3\sin\left[\left(\frac{k^2\Omega_0}{R^2+k^2}\right)^2 t\right]+A_4\cos\left[\left(\frac{k^2\Omega_0}{R^2+k^2}\right)^2 t\right],$$

where A_1,A_2,A_3,A_4 are arbitrary constants. Using procedure of the variation of constant we get particular solution:

$$x^P(t)=-\,R\,D_2\,\frac{R^2+k^2}{k^2\Omega_0},\qquad y^P(t)=-\,R\,D_1\,\frac{R^2+k^2}{k^2\Omega_0}.$$

Finally, the general solution takes the form

$$x(t) = A_1 \sin\left[\left(\frac{k^2\Omega_0}{R^2+k^2}\right)^2 t\right] + A_2 \cos\left[\left(\frac{k^2\Omega_0}{R^2+k^2}\right)^2 t\right] - R\,D_2\,\frac{R^2+k^2}{k^2\Omega_0},$$

$$y(t) = A_3 \sin\left[\left(\frac{k^2\Omega_0}{R^2+k^2}\right)^2 t\right] + A_4 \cos\left[\left(\frac{k^2\Omega_0}{R^2+k^2}\right)^2 t\right] - R\,D_1\,\frac{R^2+k^2}{k^2\Omega_0},$$

where D_1, D_2 are constants of the first integrals (77). Hence the sphere on rotating table moves along elipses parameters of which depend on initial conditions.

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