Trace decompositions of tensor spaces

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Abstract. The trace decomposition theory of tensor spaces, based on duality, is presented. The trace decomposition equations for tensors, symmetric in some sets of superscripts, and antisymmetric in the subscripts, are derived by means of the trace operations and appropriate symmetrizations and antisymmetrizations; commutation relations for the corresponding linear operators are derived. Trace decompositions of various concrete tensor spaces are discussed.

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1. Introduction

The trace decomposition of tensors on an $n$-dimensional vector space $E$, endowed with a metric tensor, belongs to classical topics of the representation theory of the orthogonal group (see Weyl [20], Fulton and Harris [5]). According to this theory, one can decompose any tensor over $E$ by means of the trace operation. Due to the presence of the metric tensor, the trace operations are defined not only in spaces of mixed tensors, but also in spaces of covariant, or contravariant tensors. Independent components in such a decomposition are traceless tensors, combined with the Kronecker $\delta$-tensor. Further relevant information on this representation theory, as well as references, can be found e.g. in Hamermesh [8], Chapter 10, and in Welsh [19] (see also Gallot, Hulin, Lafontaine [6], Chapter III, K, and Naimark [17]).

However, this metric trace decomposition theory cannot be applied to vector spaces which do not carry a metric tensor.

On the other hand, the well-known natural (i.e., $GL_n(\mathbb{R})$-equivariant) trace operation, defined on the basis of duality of vector spaces, does not require any additional structure on the underlying vector space $E$. Conceptually, this operation differs from the metric one: It is defined only between the covariant and contravariant indices of a tensor, and is invariant with respect to the general linear group. In particular, the number of traces of a tensor of type $(r, s)$ (i.e., $rs$), is in general smaller than the number of traces of a covariant tensor of type $(0, r + s)$ (i.e., $(1/2)(r + s)(r + s + 1)$).
This paper is devoted to the trace decomposition theory of mixed tensors over a real vector space $E$, based on the concept of duality. In such a setting, the trace decomposition problem belongs to the theory of systems of linear equations, or the theory of natural projectors (Krupka [10], Krupka and Janyska [16]) rather than to the group representation theory. We give a survey of elementary concepts and those recent results, which have already been applied in differential geometry and the calculus of variations in fibered spaces (Krupka [10], [13], [14]). Our main goal is to present the trace decomposition equations for some types of tensors, and to discuss the solution methods for these equations.

In Section 2 we consider tensor spaces $T^r_s E$ of type $(r, s)$ over a real, $n$-dimensional vector space $E$. We recall some standard definitions, and introduce the trace mappings and the Kronecker tensors. The duality between $T^r_s E$ and $T^s_r E$ is also studied. In Section 3 we present general trace decomposition theorems; the proofs are based on a modification of the classical Weyls method; changes are forced by the absence of the metric tensor on $E$.

In Section 4 we give an analysis of important special cases of the trace decomposition. These examples correspond with the torsion and curvature tensors studied in differential, and Finsler geometry (see e.g. Chern, Chen, Lam [3], Eisenhart [4], Gromoll, Klingenberg, Meyer [7]). It can be shown, in particular, that the well-known Weyl tensors, discovered on the basis of projective and conformal invariance (see Bokan [2], Eisenhart [4], Thomas [18], Weyl [21]), coincide with the traceless components of tensors of type $(1, 3)$, and $(2, 2)$ (Krupka [13], [14]). Our approach, represented by the trace decomposition formula, should be compared with the $O(n)$-irreducible decomposition of tensors of type $(1, 3)$ (Gallot, Hulin, Lafontaine, [6], Chapter III, K). Moreover, these examples clarify the non-uniqueness of the trace decomposition (Krupka [14], Kovár [9]).

Next sections are devoted to a difficult problem of finding the general trace decomposition formula. We discuss this problem for tensor spaces, describing underlying structures of global variational analysis, i.e., the spaces of differential forms on higher jet prolongations of fibered manifolds (Anderson [1], Krupka [11], [12], [15]). In Section 5 we give the complete trace decomposition formula for tensors, symmetric in contravariant, and antisymmetric in covariant indices (Krupka [12]). Section 6 is concerned with elementary symmetrization operators and their spectral properties, which allow us to extend the trace decomposition equations to multisymmetric-antisymmetric tensors. In Section 7 we find a suitable collection of operators, and commutation relations between them, giving us a method of solving these equations. Finally, we derive the complete trace decomposition formula for tensors, symmetric in two sets of contravariant indices, and antisymmetric in covariant indices (Section 8).

2. Tensor spaces

Throughout this paper, $\mathbb{R}$ is the field of real numbers, $E$ is a real, $n$-dimensional vector space, and $E^*$ is its dual. The dual basis to a basis $(e_1, e_2, \ldots, e_n)$ of $E$ is denoted $(e^1, e^2, \ldots, e^n)$. In a fixed basis $e_i$, a tensor $U \in T^r_s E$ is sometimes denoted by its components; we write $U = U^{i_1 i_2 \cdots i_r} e_{i_1} e_{i_2} \cdots e_{i_r}$. The Kronecker $\delta$-symbols $\delta^i_j$ and $\delta_{ij}$ are defined to be $0 \in \mathbb{R}$ if $i \neq j$ and $1 \in \mathbb{R}$ if $i = j$.

By a tensor of type $(r, s)$ over $E$, where $r$, $s$ are non-negative integers, we mean a multilinear mapping $U : E^* \times E^* \times \cdots \times E^* \times E \times \cdots \times E \rightarrow \mathbb{R}$ ($r$ factors $E^*$, $s$ factors $E$). A tensor of type $(r, 0)$ (respectively $(0, s)$) is also said to be contravariant (respectively covariant) of type $r$ (respectively $s$). The vector space of tensors of type $(r, s)$ over $E$ is denoted by $T^r_s E$. $T^0_0 E$ (respectively $T^1_0 E$) can be canonically identified with $E$ (respectively $E^*$).
To describe different tensor spaces, we use multi-indices $J = (j_1, j_2 \ldots j_r)$, where $r \geq 1$ and $1 \leq j_1, j_2, \ldots, j_r \leq n$; the lengths $|J| = r$ of different multi-indices do not necessarily coincide. We consider tensors $U = U^{j_1 j_2 \ldots j_p}_{i_1 i_2 \ldots i_s}$, symmetric in the superscripts entering each multi-index $J_k$, and antisymmetric in the subscripts; the characteristic of $U$, $\text{char} U$, is defined to be the $(p+1)$-tuple $(r_1, r_2, \ldots, r_p; s)$. The vector subspace of tensors of characteristic $(r_1, r_2, \ldots, r_p; s)$ in $T^{r_1 + r_2 + \ldots + r_p}_{s+1}$ is denoted $Z^{r_1, r_2, \ldots, r_p}_{s+1}$.

Tensors belonging to the tensor space $Z^{(r_1, r_2 \ldots r_p)}_{s+1}$ are sometimes called $p$-multi-symmetric-antisymmetric. If $p = 1$, we call these tensors symmetric-antisymmetric. If $s = 0$, we denote the corresponding tensor space by $Z^{(r_1, r_2 \ldots r_p)}_{s+1}$, and speak of $p$-multi-symmetric tensors. For 2-multi-symmetric-symmetric tensors, and for symmetric-antisymmetric tensors it is usually convenient to use the standard, explicit index notation.

If $U \in T^r_s E$ and $V \in T^p_q E$, the tensor product $U \otimes V \in T^{r+p}_{s+q} E$ is defined by

$$(U \otimes V)(\omega^1, \omega^2, \ldots, \omega^{r+p}, \xi_1, \xi_2, \ldots, \xi_{s+q}) = U(\omega^1, \omega^2, \ldots, \omega^r, \xi_1, \xi_2, \ldots, \xi_s) V(\omega^{r+1}, \omega^{r+2}, \ldots, \omega^{r+p}, \xi_{s+1}, \xi_{s+2}, \ldots, \xi_{s+q})$$

for all $\omega^\alpha \in E^*$, $\xi_\beta \in E$. The tensor $\delta = e_m \otimes e^m = \delta^i_j e_i \otimes e^j$ is the Kronecker $\delta$-tensor. If $e_i$ is a basis of $E$, then the tensors $e_{j_1} \otimes e_{j_2} \otimes \ldots \otimes e_{j_s} \otimes e^{i_1} \otimes e^{i_2} \otimes \ldots \otimes e^{i_s}$, $1 \leq i_1, i_2, \ldots, i_s, j_1, j_2, \ldots, j_r \leq n$, form the associated basis of the vector space $T^r_s E$.

Let $r$ and $s$ be positive integers, let $\alpha$ and $\beta$ be integers such that $1 \leq \alpha \leq r$, $1 \leq \beta \leq s$, and let $e_i$ be a basis of $E$. Let $U \in T^r_s E$, $U = U^{j_1 j_2 \ldots j_r}_{i_1 i_2 \ldots i_s}$, be a tensor of type $(r, s)$. We define a tensor $\text{tr}^\alpha_r U \in T^{r-1}_{s-1} E$ by

$$\text{tr}^\alpha_r U = V^{j_1 j_2 \ldots j_r}_{i_1 i_2 \ldots i_s} e_{j_1} \otimes e_{j_2} \otimes \ldots \otimes e_{j_{r-1}} \otimes e^{i_1} \otimes e^{i_2} \otimes \ldots \otimes e^{i_{s-1}},$$

where

$$V^{j_1 j_2 \ldots j_r}_{i_1 i_2 \ldots i_s} = U^{j_1 j_2 \ldots j_r}_{j_{r+1} j_{r+2} \ldots j_{r+s}} e_{j_1} \otimes e_{j_2} \otimes \ldots \otimes e_{j_{r-1}} \otimes e_{j_{r+1}} \otimes e_{j_{r+2}} \otimes \ldots \otimes e_{j_{r+s}}.$$

This tensor does not depend on the choice of the basis $e_i$. $\text{tr}^\alpha_r U$ is the $(\alpha, \beta)$-trace of $U$; the mapping $\text{tr}^\alpha_r : T^r_s E \rightarrow T^{r-1}_{s-1} E$ is the $(\alpha, \beta)$-trace mapping. A tensor $U \in T^r_s E$ is called traceless, if $\text{tr}^\alpha_r U = 0$ for all $\alpha, \beta$.

Let $V \in T^{r-1}_{s-1} E$. We define a tensor $\text{tr}^\beta_s V \in T^r_s E$ by

$$\text{tr}^\beta_s V = V^{j_1 j_2 \ldots j_{r-1}}_{i_1 i_2 \ldots i_{s-1}} e_{j_1} \otimes e_{j_2} \otimes \ldots \otimes e_{j_{r-1}} \otimes e^{i_1} \otimes e^{i_2} \otimes \ldots \otimes e^{i_{s-1}}.$$

This tensor is independent of the choice of $e_i$. The mapping $\text{tr}^\beta_s : T^{r-1}_{s-1} E \rightarrow T^r_s E$ is called the $(\alpha, \beta)$-canonical injection.

A tensor $U \in T^r_s E$ belonging to the vector subspace generated by the subspaces $\text{tr}^\alpha_r(T^{r-1}_{s-1} E) \subset T^r_s E$, where $1 \leq \alpha \leq r$, $1 \leq \beta \leq s$, is said to be a Kronecker (or $\delta$-generated) tensor. It follows from the definition that the components of a Kronecker tensor are expressible in the form of linear combinations of terms, containing the Kronecker $\delta$-symbol.

The canonical pairing of the vector spaces $T^r_s E$ and $T^r_s E$ is the bilinear mapping $T^r_s E \times T^r_s E \ni (U, V) \mapsto \langle U, V \rangle \in \mathbb{R}$ defined as follows: If in a basis of $E$,

$$U = U^{j_1 j_2 \ldots j_r}_{i_1 i_2 \ldots i_s}, \quad V = V^{k_1 k_2 \ldots k_r}_{l_1 l_2 \ldots l_r},$$

then

$$\langle U, V \rangle = U^{j_1 j_2 \ldots j_r}_{i_1 i_2 \ldots i_s} V^{l_1 l_2 \ldots l_r}_{j_1 j_2 \ldots j_r}.$$
It is easily seen that this expression is independent of the choice of \(e_i\).

Tensors \(U \in T^r E\) and \(V \in T^s E\) are said to be orthogonal, if \(\langle U, V \rangle = 0\). The orthogonal subspace to a set \(P \subset T^r E\) is the vector subspace \(Q\) of all vectors \(V \in T^s E\) satisfying \(\langle U, V \rangle = 0\) for every \(U \in P\).

The structure of tensor spaces allows us to induce isomorphisms between tensor spaces \(T^r E\) and \(T^s E\) by means of isomorphisms of \(E\) and \(E^*\). We construct these induced isomorphisms from symmetric regular bilinear forms on \(E\). Let us recall basic definitions. Let \(g \in T^2_0 E\). We say that \(g\) is symmetric, if \(g(\xi, \zeta) = g(\zeta, \xi)\) for all \(\xi, \zeta \in E\). For every \(\zeta \in E\), define a linear form \(g_\zeta \in E^*\) as the mapping \(E \ni \xi \mapsto g(\xi, \zeta) = g(\xi, \zeta) \in \mathbb{R}\). We say that \(g\) is regular, if the mapping \(\zeta \mapsto g_\zeta(\zeta) \in E^*\) is a linear isomorphism; in this case the inverse linear isomorphism of \(E^*\) into \(E\) is denoted by \(g_0^*.\)

\(g_0^*\) is immediately extended to a linear isomorphism \(g_s^r : T^r E \rightarrow T^s E\). We set for every \(U \in T^r E\), and all \(\omega^1, \omega^2, \ldots, \omega^r \in E^*\), \(\xi_1, \xi_2, \ldots, \xi_r \in E\),

\[
g_s^r U(\omega^1, \omega^2, \ldots, \omega^s, \xi_1, \xi_2, \ldots, \xi_r) = g_01\xi_1 g_01\xi_2 \ldots g_01\xi_r g_0^0\omega^1 g_0^0\omega^2 \ldots g_0^0\omega^s.
\]

\(g_s^r\) is said to be generated by \(g\). Note that while \(U\) contains \(r\) covector arguments and \(s\) vector arguments, \(g_s^r U\) depends on \(s\) covectors and \(r\) vectors.

Let in a basis \(e_i\) of \(E\), \(g = g_{ij} e^i \otimes e^j\). Since the matrix \(g_{ij}\) is regular, the inverse matrix \(g^{ij}\) satisfies \(g_{ij} g^{jk} = \delta^k_i\). Let \(\xi, \zeta \in E\), \(\omega, \eta \in E^*\). Write \(\xi = \xi^p e_p\), \(\zeta = \zeta^p e_p\), \(\omega = \omega^i e_i\), and \(\eta = \eta^r e^r\); then \((g_{ij}^0(\xi)(\zeta) = g_{ij} \xi^i \zeta^j(\xi)\) and \((g_{ij}^0(\omega)(\eta) = g^{ij} \omega^i \eta^j(\eta)\). Consequently, \(g_{ij}^0 \omega = g^{ij} \omega_i e_j\), \(g_{ij}^0 \xi = g_{ij} \xi^k e^k\), and

\[
g_s^r U = g^{k1} g^{k2} \ldots g^{k1} g_{i1} j1 g_{i2} j2 \ldots g_{il} j_l U^{j1} j2 \ldots j_r l_1 l_2 \ldots l_s
\]

\[
\cdot e_{k1} \otimes e_{k2} \otimes \ldots \otimes e_{k_s} \otimes e^{l1} \otimes e^{l2} \otimes \ldots \otimes e^{l_r},
\]

In the following lemmas we collect elementary properties of bilinear forms on tensor spaces. The proofs are immediate consequences of definitions.

**Lemma 1.** (a) For any regular, symmetric bilinear form \(g\) on \(E\),

\[
\text{tr} g_s^r \circ g_s^r = g_s^{r-1} \circ \text{tr}_r, \quad g_s^r \circ i_{\beta}^r = i^s_{\alpha} \circ g_s^{r-1}.
\]

(b) Let \(U \in T^r E\). Then \(\text{tr} g_s^r U = 0\) if and only if \(\text{tr} g_s^r U = 0\). In particular, \(U\) is traceless if and only if \(g_s^r U\) is traceless.

(c) \(U \in T^r E\) is a Kronecker tensor if and only if \(g_s^r U\) is a Kronecker tensor.

Every scalar product \(g\) on \(E\) induces a scalar product on \(T^s E\). Namely, using the same notation as for the scalar product on \(E\), we set for all \(U, V \in T^r E\), \(g(U, V) = \langle U, g_s^r V \rangle\). In a basis

\[
g(U, V) = g_{j1} k_1 g_{j2} k_2 \ldots g_{jr} k_r g^{i1} l_1 g^{i2} l_2 \ldots g^{is} l_s U^{j1} j2 \ldots j_r l_1 l_2 \ldots l_s V^{k1} k_2 \ldots k_r l_1 l_2 \ldots l_s.
\]

**Lemma 2.** Formula (2) defines a scalar product on \(T^s E\).

If \(g\) is a scalar product on \(E\), then formula (2) allows us to represent the canonical pairing \(\langle U, V \rangle \mapsto \langle U, V \rangle\) in terms of \(g\). Writing \(V = g_s^r g_s^r V\) we get

\[
\langle U, V \rangle = \langle U, g_s^r g_s^r V \rangle = g(U, g_s^r V).
\]

This expression is independent of the choice of \(g\). In particular, formula (3) shows that \(U \in T^r E\) and \(V \in T^s E\) are orthogonal if and only if \(U\) and \(g_s^r V\) are orthogonal with
respect to the scalar product (2).

Let \( \sigma \) (respectively \( \tau \)) be a permutation of the set \( \{1, 2, \ldots, r\} \) (respectively \( \{1, 2, \ldots, s\} \)), and let \( \xi_1, \xi_2, \ldots, \xi_s \in E \) (respectively \( \omega^1, \omega^2, \ldots, \omega^r \in E^* \)) be any vectors (respectively covectors). For every \( U \in T^r_s E \) we define a tensor \( (\sigma, \tau)U \in T^r_s E \) by

\[
(\sigma, \tau)U(\omega^1, \omega^2, \ldots, \omega^r, \xi_1, \xi_2, \ldots, \xi_s) = U(\omega^{\sigma(1)}, \omega^{\sigma(2)}, \ldots, \omega^{\sigma(r)}, \xi_{\tau(1)}, \xi_{\tau(2)}, \ldots, \xi_{\tau(s)}).
\]

The linear mapping \( T^r_s E \ni U \to (\sigma, \tau)U \in T^r_s E \) is called a permutation of \( T^r_s E \). In a basis, \( (\sigma, \tau)U \) has an expression

\[
(\sigma, \tau)U = U^{i_1 \sigma(1) i_2 \sigma(2) \ldots i_r \sigma(r)}_{k_{\tau(1)} k_{\tau(2)} \ldots k_{\tau(s)}} \epsilon_{i_1} \otimes \epsilon_{i_2} \otimes \cdots \otimes \epsilon_{i_r} \otimes \epsilon^{k_1} \otimes \epsilon^{k_2} \otimes \cdots \otimes \epsilon^{k_s}.
\]

**Lemma 3.** Let \( g \) be a scalar product on \( E \), and let \( (\sigma, \tau) \) be a permutation. Then

\[
g((\sigma, \tau)U, V) = g(U, (\sigma^{-1}, \tau^{-1})V).
\]

**Corollary 1.** (a) For every scalar product \( g \) on \( E \), \( (\sigma, \tau) \) is an orthogonal transformation of the vector space \( T^r_s E \), i.e., \( g((\sigma, \tau)U, (\sigma, \tau)V) = g(U, V) \).

(b) If \( \sigma \) and \( \tau \) are transpositions, then \( (\sigma, \tau) \) is a symmetric transformation.

### 3. The trace decomposition

In this section we suppose we have a fixed basis of \( E \). Recall that a tensor \( U \in T^r_s E \) is said to be traceless, if its traces are all zero, i.e.,

\[
U^{mj_1 i_1 \ldots i_r}_{mj_2 k_2 \ldots k_s} = 0, \quad U^{mj_1 i_1 \ldots i_r}_{mk_1 k_1 \ldots k_s} = 0, \ldots,
\]

\[
U^{mj_1 i_1 \ldots i_r}_{m1k_2 k_2 \ldots k_s-1 m} = 0,
\]

\[
U^{mj_1 i_1 \ldots i_r}_{mk_1 k_2 k_3 \ldots k_s-1 m} = 0,
\]

\[
\ldots
\]

\[
U^{mj_1 i_1 \ldots i_r-1 m}_{mk_2 k_3 \ldots k_s} = 0, \quad U^{mj_1 i_1 \ldots i_r-1 m}_{mk_1 k_3 k_4 \ldots k_s} = 0, \ldots,
\]

\[
U^{mj_1 i_1 \ldots i_r-1 m}_{mk_1 k_2 k_{s-1} m} = 0.
\]

A tensor \( V \in T^r_s E \) is said to be a Kronecker tensor, if there exist tensors \( V_{(q)}^{(p)} \in T^{r-1}_{s-1} E \), where \( 1 \leq p \leq r, 1 \leq q \leq s \), such that \( V \) can be expressed in the form

\[
V^{k_1 k_2 \ldots k_r l_1 l_2 \ldots l_s} = \delta^{k_1}_{l_1} V^{(1)k_2 k_3 \ldots k_r}_{l_2 l_3 \ldots l_s} + \delta^{k_1}_{l_2} V^{(1)k_2 k_3 \ldots k_r}_{l_1 l_3 \ldots l_s}
\]

\[
+ \ldots + \delta^{k_1}_{l_s} V^{(1)k_2 k_3 \ldots k_r}_{l_1 l_2 \ldots l_{s-1}} + \delta^{k_2}_{l_1} V^{(2)k_1 k_3 \ldots k_r}_{l_2 l_3 \ldots l_s}
\]

\[
+ \ldots + \delta^{k_2}_{l_2} V^{(2)k_1 k_3 \ldots k_r}_{l_1 l_3 \ldots l_s} + \ldots + \delta^{k_2}_{l_s} V^{(2)k_1 k_3 \ldots k_r}_{l_1 l_2 \ldots l_{s-1}}
\]

\[
+ \ldots + \delta^{k_r}_{l_1} V^{(r)k_1 k_2 \ldots k_{r-1}}_{l_2 l_3 \ldots l_s} + \delta^{k_r}_{l_2} V^{(r)k_1 k_2 \ldots k_{r-1}}_{l_1 l_3 \ldots l_s}
\]

\[
+ \ldots + \delta^{k_r}_{l_s} V^{(r)k_1 k_2 \ldots k_{r-1}}_{l_1 l_2 \ldots l_{s-1}}.
\]

The following result is the **trace decomposition theorem**.
Theorem 1. The vector space $T^r_s E$ is the direct sum of its subspaces of traceless and Kronecker tensors.

Proof. Existence and uniqueness can be proved by means of a scalar product on $T^r_s E$ (Section 2, Lemma 2). In the scalar product, the subspaces of traceless and Kronecker tensors become orthogonal. □

Theorem 1 can be rephrased more explicitly as follows.

Corollary 2. Let $n, r, s$ be positive integers, and let $W \in T^r_s E$, $W = W^{i_1 i_2 \ldots i_r k_1 k_2 \ldots k_s}$. There exist a unique traceless tensor $U \in T^r_s E$, and tensors $V^{(p)}_{(q)} \in T^{r-1}_{s-1} E$, where $1 \leq p \leq r$, $1 \leq q \leq s$, such that

$$W^{i_1 i_2 \ldots i_r k_1 k_2 \ldots k_s} = U^{i_1 i_2 \ldots i_r i_1 i_2 \ldots i_r} + \delta^{i_1}_{i_1} V^{(1)}_{(1)} t_{i_1} t_{i_2} \ldots t_{i_r} + \delta^{i_2}_{i_2} V^{(1)}_{(2)} t_{i_2} t_{i_3} \ldots t_{i_r} + \ldots + \delta^{i_s}_{i_s} V^{(1)}_{(s)} t_{i_1} t_{i_2} \ldots t_{i_{s-1}} + \delta^{i_1}_{i_1} V^{(2)}_{(1)} t_{i_1} t_{i_3} \ldots t_{i_r} + \delta^{i_2}_{i_2} V^{(2)}_{(2)} t_{i_2} t_{i_3} \ldots t_{i_r} + \ldots + \delta^{i_r}_{i_r} V^{(2)}_{(r)} t_{i_2} t_{i_3} \ldots t_{i_{r-1}} + \delta^{i_1}_{i_1} V^{(3)}_{(1)} t_{i_1} t_{i_3} \ldots t_{i_{r-1}} + \ldots + \delta^{i_r}_{i_r} V^{(3)}_{(r)} t_{i_1} t_{i_2} \ldots t_{i_{r-1}} + \delta^{i_1}_{i_1} V^{(4)}_{(1)} t_{i_1} t_{i_2} \ldots t_{i_{r-1}} + \ldots + \delta^{i_r}_{i_r} V^{(4)}_{(r)} t_{i_1} t_{i_2} \ldots t_{i_{r-1}} + \delta^{i_1}_{i_1} V^{(5)}_{(1)} t_{i_1} t_{i_2} \ldots t_{i_{r-1}} + \ldots + \delta^{i_r}_{i_r} V^{(5)}_{(r)} t_{i_1} t_{i_2} \ldots t_{i_{r-1}}$$

(3)

Formula (3) is the trace decomposition formula for the tensor $W$. (3) can also be viewed as the trace decomposition equations for the unknown tensors $U$ and $V^{(p)}_{(q)}$.

Remark 1. The direct sum in Theorem 1, or Corollary 1, is independent of the choice of the auxiliary metric tensor $g$.

Remark 2. Both the traceless component $U$ and the complementary Kronecker component in (3) are unique. However, this fact does not imply, in general, the uniqueness of the tensors $V^{(p)}_{(q)}$ (see Section 4).

We can apply the trace decomposition formula (3) to each of the tensors $V^{(p)}_{(q)} \in T^{r-1}_{s-1} E$. Repeating this step as many times as possible, we get the complete trace decomposition of $U$. To formulate the result more precisely, it is convenient to use explicit expressions. We need a specific summation convention allowing us to generalize formula (3). Suppose for example that $r \leq s$. Let $p$ and $r$ be fixed integers such that $1 \leq l \leq p$. By an $l$-partition of the set $\{1, 2, \ldots, l, l + 1, \ldots, p\}$ we mean two ordered subsets $\lambda_{(l,p)} = \{\lambda_1, \lambda_2, \ldots, \lambda_l\}$ and $\Lambda_{(l,p)} = \{\Lambda_{l+1}, \Lambda_{l+2}, \ldots, \Lambda_p\}$ of the set $\{1, 2, \ldots, p\}$ such that $\lambda_{(l,p)} \cap \Lambda_{(l,p)} = \emptyset$, $\lambda_{(l,p)} \cup \Lambda_{(l,p)} = \{1, 2, \ldots, l, l + 1, \ldots, p\}$. Thus, we have the inequalities $\lambda_1 < \lambda_2 < \ldots < \lambda_l$, and $\lambda_{l+1} < \lambda_{l+2} < \ldots < \lambda_p$, and it is clear that to define an $l$-partition it is sufficient to choose one of the sets $\lambda_{(l,p)}$, $\Lambda_{(l,p)}$.

Let $W \in T^r_s E$, $W = W^{i_1 i_2 \ldots i_r k_1 k_2 \ldots k_s}$, and let $l$ be an integer such that $1 \leq l \leq r$. $W$ is said to be $\delta^{(l)}$-generated, if it has an expression of the form

$$W^{i_1 i_2 \ldots i_r k_1 k_2 \ldots k_s} = \sum_{\lambda_{(l,r)}} \sum_{\xi_{(l,s)}} \sum_{\sigma \in \Sigma_l} \delta^{i_{\lambda_1}}_{k_{\sigma(1)}} \delta^{i_{\lambda_2}}_{k_{\sigma(2)}} \ldots \delta^{i_{\lambda_l}}_{k_{\sigma(l)}} V^{(\lambda_1, \lambda_2, \ldots, \lambda_l)}_{i_{\lambda_1 i_{\lambda_2} \ldots i_{\lambda_l}} k_{\sigma(1)} k_{\sigma(2)} \ldots k_{\sigma(l)}}$$

(4)
where the tensor
\[
V^r(\lambda_1, \lambda_2, \ldots, \lambda_l) = \sum_{i_1 \leq \ldots \leq i_{r-l}} k_1 k_2 \ldots k_{r-l}
\]
\[
V^r(\varphi_{\sigma(1)}, \varphi_{\sigma(2)}, \ldots, \varphi_{\sigma(l)} ) = V^r(\varphi_{\sigma(1)}, \varphi_{\sigma(2)}, \ldots, \varphi_{\sigma(l)} )
\]
belongs to $T_{s-l} E$. Each term in formula (4) is uniquely determined by two sets $\lambda_{l(r)} \subset \{1, 2, \ldots, l, l+1, \ldots, r\}$ and $\xi_{l(s)} \subset \{1, 2, \ldots, l, l+1, \ldots, s\}$, and by an element $\sigma$ of the permutation group $\Sigma_l$ of the set \{1, 2, \ldots, l\}. Note that (4) is a formal expression for the sum of terms in which all possible products of $l$ Kronecker $\delta$-tensors appear, with superscripts from the set \{1, 2, \ldots, l\} and subscripts from the set \{k_1, k_2, \ldots, k_s\}.

Obviously, $\delta^{(1)}$-generated tensors coincide with the Kronecker (i.e. $\delta$-generated) tensors, which have already been defined. A $\delta^{(l)}$-generated tensor $W \in T^r_s E$ is called $l$-
primitive, if it has a representation (4) in which all the tensors (5) are traceless; then the tensors (4) are called $\delta^{(l)}$-components of $W$. The vector subspace of $T^r_s E$ formed by $l$-primitive tensors is called $l$-
primitive.

**Remark 3.** The structure of $\delta^{(l)}$-generated tensors can be described by means of the permutation operators. Clearly, every summand in (4) is of the form $U = (\sigma, \tau)(\delta \otimes \delta \otimes \ldots \otimes \delta \otimes V)$ ($l$ factors the Kronecker $\delta$-tensors), where $\sigma$ (respectively $\tau$) is a permutation of the set \{1, 2, \ldots, r\} (respectively \{1, 2, \ldots, s\}), and $V \in T_{s-l} E$.

The following is the complete trace decomposition theorem.

**Theorem 2.** The vector space $T^r_s E$ is the direct sum of $l$-primitive subspaces.

**Proof.** From Theorem 1 it follows that every tensor $W \in T^r_s E$ can be expressed as the sum of $l$-primitive tensors, where $l = 0, 1, 2, \ldots, \min(r, s)$. We want to show that if $W$ is $l$-primitive and $m$-primitive, and $l \neq m$, then $W = \{0\}$.

Let $U \in T^r_s E$ (respectively $V \in T^r_s E$) be $l$-primitive (respectively $m$-primitive), and suppose for instance that $m < l$. We show that in a scalar product $g$ on $E$, $g(U, V) = 0$. It is sufficient to prove this equality for tensors of the form
\[
U^{j_1 2 \ldots j_r} k_1 k_2 \ldots k_s = \delta^j_{k_1} \delta^j_{k_2} \ldots \delta^j_{k_r} \delta^j_{k_{r+1}} \ldots \delta^j_{k_{r+s}},
\]
\[
V^{j_1 2 \ldots j_r} l_1 l_2 \ldots l_s = \delta^{j_1}_{l_1} \delta^{j_2}_{l_2} \ldots \delta^{j_r}_{l_r} \delta^{j_{r+1}}_{l_{r+1}} \ldots \delta^{j_{r+s}}_{l_{r+s}},
\]
determined by some partitions
\[
\lambda_{l(r)} = \{\lambda_1, \lambda_2, \ldots, \lambda_l\}, \quad \Lambda_{l(r)} = \{\Lambda_{l+1}, \Lambda_{l+2}, \ldots, \Lambda_r\},
\]
\[
\xi_{l(s)} = \{\xi_1, \xi_2, \ldots, \xi_l\}, \quad \varrho_{l(s)} = \{\varrho_{l+1}, \varrho_{l+2}, \ldots, \varrho_s\},
\]
and
\[
\theta_{m(r)} = \{\theta_1, \theta_2, \ldots, \theta_m\}, \quad \Theta_{m(r)} = \{\Theta_{m+1}, \Theta_{m+2}, \ldots, \Theta_r\},
\]
\[
\pi_{m(s)} = \{\pi_1, \pi_2, \ldots, \pi_m\}, \quad \Pi_{m(s)} = \{\Pi_{m+1}, \Pi_{m+2}, \ldots, \Pi_r, \Pi_{r+1}, \ldots, \Pi_s\},
\]

and
\[
\frac{\lambda_{l(r)}}{\Lambda_{l(r)}} \cup \frac{\xi_{l(s)}}{\varrho_{l(s)}} \cup \frac{\theta_{m(r)}}{\Theta_{m(r)}} \cup \frac{\pi_{m(s)}}{\Pi_{m(s)}}
\]
and by some permutations \( \sigma, \nu \in \Sigma_l \). We have by Section 2, (2)

\[
g(U, V) = g_{j_1 i_1} g_{j_2 i_2} \ldots g_{j_l i_l} g^{k_1 l_1} g^{k_2 l_2} \ldots g^{k_s l_s}
\]

(6)

Note that (6) includes contractions in \( i_\lambda \), \( i_\lambda \), \( i_\lambda \), and in \( l_\sigma \), \( l_\sigma \), \( l_\sigma \). Contracting the expression

\[
g_{j_1 i_1} g_{j_2 i_2} \ldots g_{j_l i_l} g^{k_1 l_1} g^{k_2 l_2} \ldots g^{k_s l_s} \delta_{k_1}^{i_\lambda_1} \delta_{k_2}^{i_\lambda_2} \ldots \delta_{k_{l_\sigma}}^{i_\lambda_{l_{\sigma}}} U(\lambda_1, \lambda_2, \ldots, \lambda_l) i_{\lambda_{l+1}} i_{\lambda_{l+2}} \ldots i_{\lambda_r}
\]

\[
\delta_{l_{\sigma_1}}^{j_{\rho_1}} \delta_{l_{\sigma_2}}^{j_{\rho_2}} \ldots \delta_{l_{\sigma_{l_{\sigma}}}}^{j_{\rho_{l_{\sigma}}}} V(\theta_1, \theta_2, \ldots, \theta_m) j_{\sigma_{m+1}} j_{\sigma_{m+2}} \ldots j_{\sigma_r}
\]

in \( i_\lambda \), \( i_\lambda \), \( i_\lambda \) yields

\[
g_{j_1 i_1} g_{j_2 i_2} \ldots g_{j_l i_l} g^{k_1 l_1} g^{k_2 l_2} \ldots g^{k_s l_s} \delta_{k_1}^{i_\lambda_1} \delta_{k_2}^{i_\lambda_2} \ldots \delta_{k_{l_\sigma}}^{i_\lambda_{l_\sigma}} = \delta_{j_{\rho_1}}^{l_{\sigma_1}} \delta_{j_{\rho_2}}^{l_{\sigma_2}} \ldots \delta_{j_{\rho_{l_{\sigma}}}}^{l_{\sigma_{l_{\sigma}}}}
\]

The remaining expression in (6) should be contracted in \( l_\sigma \), \( l_\sigma \), \( l_\sigma \). Contract the product

\[
\delta_{l_{\sigma_1}}^{j_{\rho_1}} \delta_{l_{\sigma_2}}^{j_{\rho_2}} \ldots \delta_{l_{\sigma_{l_{\sigma}}}}^{j_{\rho_{l_{\sigma}}}}
\]

in any possible way. Since \( m < l \), the result of the contraction always contains at least one factor \( \delta_{j_{\rho_1}}^{l_{\sigma_1}} \). This factor, however, should be contracted with

\[
V(\theta_1, \theta_2, \ldots, \theta_m) j_{\sigma_{m+1}} j_{\sigma_{m+2}} \ldots j_{\sigma_r} l_{\sigma_{m+1}} l_{\sigma_{m+2}} \ldots l_{\sigma} l_{\sigma+1} \ldots l_{\sigma_{l_{\sigma}}},
\]

which results in a trace of

\[
V(\theta_1, \theta_2, \ldots, \theta_m) l_{\sigma_{m+1}} l_{\sigma_{m+2}} \ldots l_{\sigma} l_{\sigma+1} \ldots l_{\sigma_{l_{\sigma}}}.
\]

But by hypothesis, this tensor is \( l \)-primitive, so its traces are all zero. This proves that \( g(U, V) = 0 \) as required.

Now if \( W \) is \( l \)-primitive and \( m \)-primitive, where \( l \neq m \), we have \( g(W, W) = 0 \) hence \( W = 0 \). This completes the proof. \( \square \)

The following is another version of the complete trace decomposition theorem.

**Corollary 3.** Let \( W \in T_s^r E \), and let \( r \leq s \). There exist unique \( l \)-primitive tensors \( W^{(l)} \in T_s^r E \), where \( 0 \leq l \leq r \), such that

\[
W = W^{(0)} + W^{(1)} + W^{(2)} + \ldots + W^{(r)}.
\]

Formula (7) is the complete trace decomposition formula. Every tensor belonging to \( T_s^r E \) can be expressed in the form (7).

We are now concerned with the uniqueness of the \( \delta^{(l)} \)-components in (7).

**Theorem 3.** Let \( W \in T_s^r E \). If \( r + s \leq n + 1 \) then all \( \delta^{(l)} \)-components of \( W \) are unique.

**Proof.** The simplest proof is an explicit one. Suppose that \( r \leq s \). We prove that if \( W^{(l)} = 0 \), \( 1 \leq l \leq r \), then the \( \delta^{(l)} \)-component of \( W \) vanishes. According to (4), we must show that the homogeneous system
\[
\sum_{\lambda_1, \ldots, \lambda_t} \sum_{\sigma \in \Sigma_t} \delta_{k_{\sigma(1)}}^{i_1} \delta_{k_{\sigma(2)}}^{i_2} \cdots \delta_{k_{\sigma(t)}}^{i_t} \cdot V^{(\lambda_1, \lambda_2, \ldots, \lambda_t)}_{(\varphi_1, \varphi_2, \ldots, \varphi_t)} \cdot i_{\lambda_{t+1}} i_{\lambda_{t+2}} \cdots i_{\lambda_r} k_{\theta_{t+1}} k_{\theta_{t+2}} \cdots k_{\theta_s} = 0
\]

has the trivial solution
\[
V^{(\omega_1, \omega_2, \ldots, \omega_t)}_{(\mu_1, \mu_2, \ldots, \mu_t)} = 0
\]
only. Using any partitions \(\varphi_{(l,r)} = \{\varphi_1, \varphi_2, \ldots, \varphi_l\}, \Phi_{(l,r)} = \{\Phi_{l+1}, \Phi_{l+2}, \ldots, \Phi_r\}, \psi_{(l,s)} = \{\psi_1, \psi_2, \ldots, \psi_l\}, \Psi_{(l,s)} = \{\Psi_{l+1}, \Psi_{l+2}, \ldots, \Psi_s\}\), and a permutation \(\mu \in \Sigma_t\), we can prove that the tensor on the left in (8) is \(\delta\)-generated. Since by hypothesis this tensor is traceless, this implies (8). \(\square\)

4. The trace decomposition formula: Special cases

We give the trace decomposition formulas for tensor spaces of types \((1, 1)\), \((1, 2)\), \((1, 3)\), \((2, 2)\), and \((2, 3)\) over \(E\). Our first example is trivial, but is presented for the record: If \(U \in T^1 E\), \(U = U^k_k\), then there exist a unique traceless tensor \(V = V^k_k\) and a unique number \(c \in \mathbb{R}\) such that \(U^k_k = V^k_k + c\delta^k_k\); \(c\) is given by \((1/n)U^r_k\). Tensors of type \((1, 4)\) were considered by Kovář by means of Maple; the resulting trace decomposition formula is very long (see [9], [14]).

Let us discuss tensors of type \((1, 2)\) (the torsion tensors in differential geometry of connections).

**Theorem 4.** Let \(U \in T^1_2 E\), \(U = U^k_i\). There exist a unique traceless tensor \(V \in T^1_2 E\), \(V = V^k_i\), and unique tensors \(P, Q \in T^0_1 E\), \(P = P^r_l\), \(Q = Q^r_l\), such that

\[
U^k_i = V^k_i + \delta^k_l P^l_i + \delta^i_l Q^r_l.
\]

These tensors are given by

\[
P^i_l = \frac{1}{n^2 - 1} (nU^r_l + U^r_l) , \quad Q^r_l = \frac{1}{n^2 - 1} (-U^r_l + nU^r_l).
\]

Note that the tensor \(V\) is already defined by the tensors \(P, Q,\) and \(U\).

Now consider tensors of type \((1, 3)\) and \((2, 2)\) (curvature tensors describing properties of connections and metric fields on smooth manifolds). Of particular importance are traceless tensors of this type, the Weyl tensors; as a consequence of the trace decomposition theory, they can be defined as traceless components of tensors of type \((1, 3)\), satisfying additional symmetry properties. For more discussion, we refer to Krupka [12], [13]. Note that the presented formulas illustrate possible non-uniqueness of the trace decomposition.

**Theorem 5.** Let \(U \in T^1_3 E\), \(U = U^i_{klm}\).

(a) Suppose that \(n \geq 3\). Then there exist a unique traceless tensor \(V \in T^1_3 E\), \(V = V^i_{klm}\), and unique tensors \(P, Q, R \in T^0_3 E\), \(P = P^r_{lm}\), \(Q = Q^r_{km}\), \(R = R^r_{kl}\), such that

\[
U^i_{klm} = V^i_{klm} + \delta^i_k P^r_{lm} + \delta^r_i Q^r_{km} + \delta^r_m R^r_{kl}.
\]
These tensors are given by

\[
P_{kl} = \frac{1}{(n^2 - 1)(n^2 - 4)} \left( n(n^2 - 3)U_{tkl}^t - (n^2 - 2)U_{klt}^t \right)
+ nU_{klt}^t - 2U_{tlt}^t + nU_{lkt}^t - (n^2 - 2)U_{lkt}^t,
\]

\[
Q_{kl} = \frac{1}{(n^2 - 1)(n^2 - 4)} \left( -(n^2 - 2)U_{tkl}^t + n(n^2 - 3)U_{klt}^t \right)
- (n^2 - 2)U_{klt}^t + nU_{lkt}^t - 2U_{lkt}^t + nU_{lkt}^t,
\]

\[
R_{kl} = \frac{1}{(n^2 - 1)(n^2 - 4)} \left( nU_{tkl}^t - (n^2 - 2)U_{klt}^t + n(n^2 - 3)U_{klt}^t \right)
- (n^2 - 2)U_{klt}^t + nU_{lkt}^t - 2U_{lkt}^t.
\]

(b) Suppose that \(n = 2\). Then there exist a unique traceless tensor \(V \in T^1_2 E, V = V_{klm}^i\), and tensors \(P, Q, R \in T^2_2 E, P = P_{kl}, Q = Q_{kl}, R = R_{kl}\), such that

\[
U_{klm}^i = V_{klm}^i + \delta_k^i P_{lm} + \delta_l^i Q_{km} + \delta_m^i R_{kl}.
\]

These tensors are given by

\[
P_{11} = \frac{1}{4} \left( 3U_{111}^t - U_{11t}^t - U_{1tt}^t \right),
\]

\[
P_{12} = \frac{5}{6} \left( U_{12t}^t - \mu \right) - \frac{2}{3} U_{12t}^t + \frac{1}{2} U_{12t}^t - \frac{1}{3} U_{21t}^t + \frac{5}{6} (U_{21t}^t - \mu),
\]

\[
P_{21} = - \frac{1}{3} \left( U_{12t}^t - \mu \right) + \frac{2}{3} U_{12t}^t - U_{12t}^t + \frac{4}{3} U_{21t}^t - \frac{2}{3} (U_{21t}^t - \mu),
\]

\[
P_{22} = \frac{1}{4} \left( 3U_{22t}^t - U_{22t}^t - U_{22t}^t \right),
\]

\[
Q_{11} = \frac{1}{4} \left( -U_{11t}^t + 3U_{11t}^t - U_{11t}^t \right),
\]

\[
Q_{12} = - \frac{2}{3} \left( U_{12t}^t - \mu \right) + \frac{4}{3} U_{12t}^t - U_{12t}^t + \frac{2}{3} U_{21t}^t - \frac{1}{3} (U_{21t}^t - \mu),
\]

\[
Q_{21} = \frac{1}{6} \left( U_{12t}^t - \mu \right) - \frac{1}{3} U_{12t}^t + \frac{1}{2} U_{12t}^t - \frac{2}{3} U_{21t}^t + \frac{5}{6} (U_{21t}^t - \mu),
\]

\[
Q_{22} = \frac{1}{4} \left( -U_{12t}^t + 3U_{22t}^t - U_{22t}^t \right),
\]

\[
R_{11} = \frac{1}{4} \left( -U_{11t}^t - U_{11t}^t + 3U_{11t}^t \right),
\]

\[
R_{12} = \frac{1}{2} \left( U_{12t}^t - \mu \right) - U_{12t}^t + \frac{3}{2} U_{12t}^t - U_{21t}^t + \frac{1}{2} (U_{21t}^t - \mu),
\]

\[
R_{21} = \mu,
\]

\[
R_{22} = \frac{1}{4} \left( -U_{12t}^t - U_{22t}^t + 3U_{22t}^t \right),
\]

where \(\mu \in \mathbb{R}\) is a parameter.
Theorem 6. Let \( U \in T^r_s E \), \( U = U^{ij}_{kl} \).

(a) Suppose that \( n \geq 3 \). Then there exist a unique traceless tensor \( V \in T^r_s E \), \( V = V^{ij}_{kl} \), unique traceless tensors \( P, Q, R, S \in T^1_1 E \), \( P = P^i_k \), \( Q = Q^i_k \), \( R = R^i_k \), \( S = S^i_k \), and unique numbers \( G, H \in \mathbb{R} \), such that

\[
U^{ij}_{kl} = V^{ij}_{kl} + \delta^i_k P^j_l + \delta^i_l Q^j_k + \delta^j_k R^i_l + \delta^j_l S^i_k + \delta^i_l \delta^j_k G + \delta^i_k \delta^j_l H.
\]

These tensors are given by

\[
P^i_j = \frac{1}{n(n^2 - 4)}((n^2 - 2)U^{si}_{sj} - nU^{is}_{sj} - nU^{js}_{sj} + 2U^{is}_{js} - n\delta^i_j U^{st}_{ts} + 2\delta^i_j U^{st}_{ts}),
\]

\[
Q^i_j = \frac{1}{n(n^2 - 4)}(-nU^{si}_{sj} + (n^2 - 2)U^{js}_{sj} - nU^{is}_{js} + 2\delta^i_j U^{st}_{st} - n\delta^i_j U^{st}_{ts}),
\]

\[
R^i_j = \frac{1}{n(n^2 - 4)}(-nU^{si}_{sj} + 2U^{is}_{js} - nU^{js}_{sj} + 2\delta^i_j U^{st}_{st} - n\delta^i_j U^{st}_{ts}),
\]

\[
S^i_j = \frac{1}{n(n^2 - 4)}(2U^{is}_{sj} - nU^{is}_{js} - nU^{js}_{sj} + (n^2 - 2)U^{is}_{js} + 2\delta^i_j U^{st}_{st} - n\delta^i_j U^{st}_{ts}),
\]

\[
G = \frac{1}{n(n^2 - 1)}(nU^{st}_{st} - U^{st}_{ts}),
\]

\[
H = \frac{1}{n(n^2 - 1)}(-U^{st}_{st} + nU^{st}_{ts}).
\]

(b) Suppose that \( n = 2 \). Then there exist a unique traceless tensor \( V \in T^r_s E \), \( V = V^{ij}_{kl} \), traceless tensors \( P, Q, R, S \in T^1_1 E \), \( P = P^i_k \), \( Q = Q^i_k \), \( R = R^i_k \), \( S = S^i_k \), and unique numbers \( G, H \in \mathbb{R} \) such that

\[
U^{ij}_{kl} = V^{ij}_{kl} + \delta^i_k P^j_l + \delta^i_l Q^j_k + \delta^j_k R^i_l + \delta^j_l S^i_k + \delta^i_l \delta^j_k G + \delta^i_k \delta^j_l H.
\]

These tensors are given by

\[
P^i_j = U^{st}_{sj} - \frac{1}{2} U^{is}_{sj} - \frac{1}{2} U^{js}_{sj} - \frac{1}{2} \delta^i_j U^{st}_{ts} + \frac{1}{2} \delta^i_j U^{st}_{ts} + \mu^i_j,
\]

\[
Q^i_j = -\frac{1}{2} U^{st}_{sj} + \frac{3}{4} U^{is}_{sj} + \frac{1}{4} U^{js}_{sj} + \frac{1}{4} \delta^i_j U^{st}_{st} - \frac{1}{2} \delta^i_j U^{st}_{ts} - \mu^i_j,
\]

\[
R^i_j = -\frac{1}{2} U^{st}_{sj} + \frac{3}{4} U^{is}_{sj} + \frac{1}{4} U^{js}_{sj} + \frac{1}{4} \delta^i_j U^{st}_{st} - \frac{1}{2} \delta^i_j U^{st}_{ts} - \mu^i_j,
\]

\[
S^i_j = \mu^i_j,
\]

\[
G = \frac{1}{6} (2U^{st}_{st} - U^{st}_{ts}),
\]

\[
H = \frac{1}{6} (-U^{st}_{st} + 2U^{st}_{ts}),
\]

where \( \mu^i_j \) are real parameters such that \( \mu^1_1 + \mu^2_2 = 0 \).

In Section 3, Theorem 3, we have shown that the condition \( r + s \leq n + 1 \) is sufficient for the uniqueness of the complete trace decomposition of the tensor space \( T^r_s E \). The following result shows, in particular, that for \( n = 3 \), this condition is not necessary. The proof consists in finding an explicit form of the trace decomposition equations, and determining their rank by a direct computation of the corresponding determinants.
Theorem 7. Let $U \in T^2_3E$, $U = U^{k_1k_2}_{j_1j_2j_3}$, and suppose that $n \geq 3$. Then there exist a unique traceless tensor $V \in T^2_3E$, $V = V^{k_1k_2}_{j_1j_2j_3}$, unique tensors $V^{(a)}_{(b)} \in T^2_3E$, where $\alpha = 1, 2$, $\beta = 1, 2, 3$, and unique tensors $V_{(2,1)}^{(1,2)}$, $V_{(1,3)}^{(1,2)}$, $V_{(2,3)}^{(1,2)}$, such that

\begin{align*}
U^{k_1k_2}_{j_1j_2j_3} &= V^{k_1k_2}_{j_1j_2j_3} + \delta^{j_1}_{j_2} V^{(1)k_2}_{j_2j_3} + \delta^{j_1}_{j_3} V^{(2)k_1}_{j_1j_2} + \delta^{j_2}_{j_3} V^{(2)k_1}_{j_1j_2} \\
&+ \delta^{j_1}_{j_3} V^{(1)k_1}_{j_1j_2} + \delta^{j_2}_{j_3} V^{(1)k_1}_{j_1j_2} + \delta^{j_1}_{j_2} V^{(1)k_2}_{j_1j_3} + \delta^{j_2}_{j_3} V^{(2)k_1}_{j_1j_3} + \delta^{j_1}_{j_3} V^{(2)k_1}_{j_1j_3} + \delta^{j_2}_{j_3} V^{(2)k_1}_{j_1j_3} \\
&+ \delta^{j_1}_{j_3} V^{(1)k_1}_{j_1j_3} + \delta^{j_2}_{j_3} V^{(1)k_1}_{j_1j_3} + \delta^{j_1}_{j_3} V^{(2)k_1}_{j_1j_3} + \delta^{j_2}_{j_3} V^{(2)k_1}_{j_1j_3} + \delta^{j_1}_{j_3} V^{(2)k_1}_{j_1j_3} + \delta^{j_2}_{j_3} V^{(2)k_1}_{j_1j_3}.
\end{align*}

5. Symmetric-antisymmetric tensors

Let $E$ be an $n$-dimensional vector space. A tensor $U \in T^r_sE$ is said to be symmetric-antisymmetric, if it is symmetric in all superscripts and antisymmetric in all subscripts. We denote the subspace of symmetric-antisymmetric tensors in $T^r_sE$ by $Z^{(r)}_{(s)}E$; we wish to present in this section the trace decomposition formula for these tensors.

Let $U \in Z^{(r)}_{(s)}E$ be a tensor. We set

$$C_{r,s} = \frac{(r+1)(s+1)}{n+r-s},$$

and

\begin{equation}
q U = C_{r,s} U^{i_1i_2\ldots i_{r+1}}_{j_1j_2\ldots j_{r+1}} \delta^{i_1}_{j_1} \text{ alt}(j_1j_2\ldots j_{s+1}) \text{ sym}(i_1i_2\ldots i_{r+1}),
\end{equation}

where alt (respectively sym) means alternation (respectively symmetrization) in the indicated indices, and

\begin{equation}
tr U = tr(1)U = U^{p_1p_2\ldots p_{r-1}}_{j_1j_2\ldots j_{r-1}}.
\end{equation}

The following two theorems summarize basic properties of the linear mappings $q : Z^{(r)}_{(s)}E \rightarrow Z^{(r+1)}_{(s+1)}E$ and $tr : Z^{(r)}_{(s)}E \rightarrow Z^{(r-1)}_{(s-1)}E$, and of linear equations associated with them.

**Theorem 8.** Every tensor $U \in Z^{(r)}_{(s)}E$ satisfies $q U = 0$, $tr U = 0$, and

\begin{equation}
U = q tr U + tr q U.
\end{equation}

**Proof.** Theorem 1 can be proved by explicit computations. □

**Theorem 9.** Let $U \in Z^{(r)}_{(s)}E$.

(a) Equation $q V + tr W = U$ for unknown tensors $V \in Z^{(r-1)}_{(s-1)}E$, $W \in Z^{(r+1)}_{(s+1)}E$ has a unique solution such that $tr V = 0$, $q W = 0$. This solution is given by $V = tr U$, $W = q U$.

(b) Equation $q X = U$ has a solution $X \in Z^{(r-1)}_{(s-1)}E$ if and only if $q U = 0$. If this
condition is satisfied, then \( X = \text{tr} \ U \) is a solution. Any other solution is of the form \( X' = X + q \ Y \) for some tensor \( Y \in \mathcal{Z}^{(r-12)}(s-2)E \).

**Remark 4.** Formula (3) is the complete trace decomposition formula of \( U \). This formula has a remarkable structure: It shows that formally, the operators \( q \) and \( \text{tr} \) have the same properties as the homotopy operator in the integrability theory of differential forms on smooth manifolds.

**Remark 5.** Equations of the form \( \text{tr} \ X = U \) can be solved in the same way as the equation \( q \ X = U \).

### 6. Elementary symmetrization operators

In this section, we use multi-indices of the form \( J = (j_1, j_2, \ldots, j_r) \), where \( r \geq 1 \), \( 1 \leq j_1, j_2, \ldots, j_r \leq n \); the length of \( J \) is defined to be \( |J| = r \). We consider contravariant tensors \( U = U^{j_1 j_2 \ldots j_r} \), symmetric in the superscripts entering each multi-index; the characteristic of \( U \), \( \text{char} \ U \), is defined to be the \( p \)-tuple \( (r_1, r_2, \ldots, r_p) \), where \( |J_i| = r_i \). The lengths of different multi-indices do not necessarily coincide. The vector subspace of tensors of characteristic \( (r_1, r_2, \ldots, r_p) \) in \( T^{r_1+r_2+\ldots+r_p}E \) is denoted \( \mathcal{Z}^{(r_1, r_2, \ldots, r_p)} \). The tensors belonging to the tensor space \( \mathcal{Z}^{(r_1, r_2, \ldots, r_p)} \) are called \( p \)-multisymmetric; if \( p = 1 \), we speak of symmetric tensors.

We introduce linear mappings between multisymmetric tensor spaces, representing successive symmetrizations in different tensor indices, and study their spectral properties. These mappings appear in the trace decomposition formulas, and will be used in next sections. It is sufficient to define them for \( 2 \)-multisymmetric tensors. In this case we usually use the standard index notation which is more explicit; we write \( U = U^{j_1 j_2 \ldots j_l} j_1^{} j_2^{} \ldots j_m^{} \) for \( 2 \)-multisymmetric tensors of characteristic \( (l, m) \), where \( l, m \geq 0 \). All formulas can be easily written for general multisymmetric tensors (i.e., for any two multi-indices labelling these tensors).

Let \( U \in \mathcal{Z}^{(l, m)} \), \( U = U^{j_1 j_2 \ldots j_l} j_1^{} j_2^{} \ldots j_m^{} \), be a tensor. We assign to \( U \) new tensors \( \text{sym}_{1,2} \ U \in \mathcal{Z}^{(l+1, m-1)} \) and \( \text{sym}_{2,1} \ U \in \mathcal{Z}^{(l-1, m+1)} \) by

\[
\text{sym}_{1,2} \ U = U^{j_1 j_2 \ldots j_l} j_1^{} j_2^{} \ldots j_m^{} j_m^{} \ldots j_{l+1}^{} \text{sym}(j_1 j_2 \ldots j_l j_{l+1}),
\]

\[
\text{sym}_{2,1} \ U = U^{j_1 j_2 \ldots j_l j_{l+1}^{} j_1^{} j_2^{} \ldots j_m^{} j_m^{} \ldots j_{l+1}^{}}, \text{sym}(k_1 k_2 \ldots k_m k_{m+1}).
\]

In the symmetrizations the usual symmetrization coefficients are included which are equal in the considered case to \( 1/(l+1) \) and \( 1/(m+1) \), respectively.

Note that \( \text{char} \ \text{sym}_{1,2} \ U = (l+1, m-1) \), \( \text{char} \ \text{sym}_{2,1} \ U = (l-1, m+1) \), and

\[
\text{char} \ \text{sym}_{1,2}^p \ U = (l+p, m-p), \quad \text{char} \ \text{sym}_{2,1}^p \ U = (l-p, m+p)
\]

whenever the operators \( \text{sym}_{1,2}^p \), \( \text{sym}_{2,1}^p \) are defined. Composing the operator \( \text{sym}_{1,2}^p \) several times, we obtain

\[
\text{sym}_{1,2}^p \ U = U^{j_1 j_2 \ldots j_m j_{m+1}} j_1^{} j_2^{} \ldots j_{m+1}^{} \text{sym}(j_1 j_2 j_3 \ldots j_l j_{l+1})
\]

Thus, the powers of \( \text{sym}_{1,2}^p \) satisfy

\[
\text{sym}_{1,2}^m \ U = U^{j_1 j_2 \ldots j_m j_{m+1}} j_1^{} j_2^{} \ldots j_{m+1}^{} \text{sym}(j_1 j_2 j_3 \ldots j_l j_{l+1} \ldots j_{l+m})
\]
We note that \( \text{sym}^l_{1,2} U \) (respectively \( \text{sym}^p_{1,2} U \)) is defined only for \( p \leq m \) (respectively \( p \leq l \)). \( \text{sym}^m_{1,2} U \) (respectively \( \text{sym}^l_{1,2} U \)) is a symmetric tensor of characteristic \((l + m, 0, s)\) (respectively \((0, m + l, s)\)). The operators \( \text{sym}^l_{1,2} \text{sym}^l_{1,2} U \) and \( \text{sym}^m_{1,2} \text{sym}^m_{1,2} U \) are equal to the complete symmetrization in the space of tensors of characteristic \((l, m)\), \(Z^{(l, m)} E\). In particular, \( \text{sym}^l_{1,2} \text{sym}^l_{1,2} \text{sym}^l_{1,2} = \text{sym}^m_{2,1} \text{sym}^m_{2,1} \text{sym}^m_{2,1} \text{sym}^m_{1,2} = \text{sym}^m_{2,1} \text{sym}^m_{1,2} \).

We also introduce slightly modified symmetrization operators, differing from (1) by a numerical factor,

\[
\sigma = (l + 1) \text{sym}^l_{1,2}, \quad \tau = (m + 1) \text{sym}^m_{2,1}.
\]

We call these operators the elementary symmetrization operators. Note that

\[
\tau^k U = \frac{(m + k)!}{m!} \text{sym}^k_{2,1} U,
\]

and

\[
\sigma^k \tau^k U = \frac{(m + k)!}{m!} \frac{l!}{(l - k)!} \text{sym}^l_{1,2} \text{sym}^l_{1,2} U.
\]

For \( k = l \), the complete symmetrization operator \( \text{sym}^l_{1,2} \text{sym}^l_{1,2} \text{sym}^l_{1,2} \) satisfies

\[
\sigma^l \tau^l U = \tau^m \sigma^m U = \frac{l!(m + l)!}{m!} \text{sym}^l_{1,2} \text{sym}^l_{1,2} U = \frac{m!(m + l)!}{l!} \text{sym}^m_{2,1} \text{sym}^m_{2,1} U.
\]

**Lemma 1.** The operators \( \sigma \) and \( \tau \) satisfy the commutation relation

\[
\tau \sigma U - \sigma \tau U = (m - l)U.
\]

**Proof.** This is an immediate consequence of an explicit formula

\[
\text{sym}^l_{2,1} \text{sym}^l_{1,2} U = \frac{1}{l + 1} U^{j_1 j_2 \ldots j_l k_1 k_2 \ldots k_m}
\]

\[
+ \frac{1}{m(l + 1)} (U^{k_1 j_1 j_2 \ldots j_l k_1 k_2 k_3 \ldots k_m} + U^{k_1 j_1 j_2 j_l j_1 \ldots j_l j_1 k_2 k_3 \ldots k_m} + U^{k_2 j_1 j_2 j_l j_1 \ldots j_l j_1 k_1 k_2 k_3 \ldots k_m} + U^{k_1 j_1 j_2 \ldots j_l j_1 k_1 k_2 k_3 \ldots k_m} + U^{k_2 j_1 j_2 \ldots j_l j_1 j_1 \ldots j_l j_1 k_1 k_2 k_3 \ldots k_m} + U^{k_1 j_1 j_2 \ldots j_l j_1 j_1 \ldots j_l j_1 k_2 k_3 \ldots k_m} + U^{k_2 j_1 j_2 \ldots j_l j_1 j_1 \ldots j_l j_1 k_2 k_3 \ldots k_m} + U^{k_1 j_1 j_2 \ldots j_l j_1 j_1 \ldots j_l j_1 k_3 k_2 \ldots k_m} + U^{k_2 j_1 j_2 \ldots j_l j_1 j_1 \ldots j_l j_1 k_3 k_2 \ldots k_m} + U^{k_1 j_1 j_2 \ldots j_l j_1 j_1 \ldots j_l j_1 k_3 k_2 \ldots k_m} + U^{k_2 j_1 j_2 \ldots j_l j_1 j_1 \ldots j_l j_1 k_3 k_2 \ldots k_m} + U^{k_1 j_1 j_2 \ldots j_l j_1 j_1 \ldots j_l j_1 k_3 k_2 \ldots k_m} + U^{k_2 j_1 j_2 \ldots j_l j_1 j_1 \ldots j_l j_1 k_3 k_2 \ldots k_m})
\]

which implies

\[
m(l + 1) \text{sym}^l_{2,1} \text{sym}^l_{1,2} U - mU^{j_1 j_2 \ldots j_l k_1 k_2 \ldots k_m} = l(m + 1) \text{sym}^m_{2,1} \text{sym}^m_{2,1} U - lU^{j_1 j_2 \ldots j_l k_1 k_2 \ldots k_m}. \]

We find relations between the powers \((\sigma \tau)^p\) and the composite operators \(\sigma^p \tau^p\). For tensors of characteristic \((l, m)\), put

\[
N_k = (k + 1)(m - l + k)
\]
for all \( k = 0, 1, 2, \ldots, l - 1 \). Note that 
\[
N_{k+1} - N_k = m - l + 2(k + 1),
\]
and
\[
2 + m - l \leq m - l + 2(k + 1) \leq m - l + 2l = m + l.
\]
If \( m \geq l \), then \( N_k \) will be an increasing sequence of positive numbers. Denote by \( \Sigma^p_q \), where \( 0 \leq p \leq l, \ 0 \leq q \leq p \), the double sequence of symmetric polynomials in the variables \( N_k \). These polynomials are defined by
\[
\Sigma^0_1 = 0, \quad \Sigma^p_0 = 1,
\]
and for every \( p, \ 1 \leq p \leq l \),
\[
\begin{align*}
\Sigma^p_1 &= \Sigma^{p-1}_0 + \Sigma^{p-1}_0 N_{p-1}, \\
\Sigma^p_2 &= \Sigma^{p-1}_1 + \Sigma^{p-1}_1 N_{p-1}, \\
&\vdots \\
\Sigma^p_{p-1} &= \Sigma^{p-1}_{p-1} + \Sigma^{p-1}_{p-1} N_{p-1}, \\
\Sigma^p_p &= \Sigma^{p-1}_{p-1} N_{p-1}.
\end{align*}
\]
Note that both \( N_k \) and \( \Sigma^p_q \) depend on the characteristic \( (l, m) \) of the underlying tensor space; in fact, \( \Sigma^p_q \) is a function of the difference \( m - l \). To express this dependence, we sometimes write
\[
N_k = N_k^{(m-l)}, \quad \Sigma^p_q = \Sigma^p_q^{(m-l),p}.
\]

**Theorem 10.** For any tensor \( U \) of characteristic \( (l, m) \), and all \( k = 0, 1, 2, \ldots, l - 1 \),
\[
\begin{align*}
\sigma^k \tau^k U &= (-1)^0 \Sigma^0_0 (\tau \sigma)^k U + (-1)^1 \Sigma^1_1 (\tau \sigma)^{k-1} U + (-1)^2 \Sigma^2_2 (\tau \sigma)^{k-2} U \\
&\quad + \ldots + (-1)^{k-1} \Sigma^{k-1}_k (\tau \sigma) U + (-1)^k \Sigma^k_k U.
\end{align*}
\]

**Proof.** From Lemma 1, (5) it follows that \( \tau^2 \sigma U - \tau \sigma^2 U = (m - l) \tau U \) and, since \( \text{char} \tau U = (l - 1, m + 1) \) and \( \tau \sigma \tau U = \sigma \tau^2 U + (m - l + 2) \tau U \), we have
\[
\tau^2 \sigma U - \tau \sigma^2 U = (m - l) \tau U + (m - l + 2) \tau U = 2(m - l + 1) \tau U.
\]
We apply \( \tau \) to (11); we get \( \tau^3 \sigma U - \tau \sigma^2 U = 2(m - l + 1) \tau^2 U \). But the characteristic \( \text{char} \tau U \) is \( \text{char} \tau U = (l - 2, m + 2) \), so we have \( \tau \sigma \tau^2 U = \sigma \tau^3 U + (m - l + 4) \tau^2 U \), and \( \tau^3 \sigma U - \tau \sigma^2 U = 3(m - l + 2) \tau^2 U \). Now it is easy to prove by induction that for all \( k = 0, 1, 2, \ldots, l \),
\[
\tau^k \sigma U - \sigma^k U = N_{k-1} \tau^{k-1} U.
\]
Apply \( \sigma^k \) to (12). We have \( \sigma^k \tau^k \sigma U - \sigma^{k+1} \tau^{k+1} U = N_k \sigma^k \tau^k U \), i.e.,
\[
\sigma^{k+1} \tau^{k+1} U = \sigma^k \tau^k (\tau \sigma U - N_k U).
\]
Using formula (13) repeatedly, we can express \( \sigma^{k+1} \tau^{k+1} \) as the sum of powers of \( \tau \sigma \). For \( k = 0, 1, 2 \) we obtain
\[
\begin{align*}
\sigma \tau U &= \tau \sigma U - \Sigma^1_1 U, \\
\sigma^2 \tau^2 U &= (\tau \sigma)^2 U - \Sigma^2_1 \tau \sigma U + \Sigma^2_2 U, \\
\sigma^3 \tau^3 U &= (\tau \sigma)^3 U - \Sigma^3_1 (\tau \sigma)^2 U + \Sigma^3_2 \tau \sigma U - \Sigma^3_3 U.
\end{align*}
\]
On induction one can prove that
\[
\sigma^k \tau^k U = (-1)^0 \Sigma_0^k (\tau \sigma)^k U + (-1)^1 \Sigma_1^k (\tau \sigma)^{k-1} U + (-1)^2 \Sigma_2^k (\tau \sigma)^{k-2} U \\
+ \ldots + (-1)^{k-1} \Sigma_{k-1}^k \tau^k \sigma U + (-1)^k \Sigma_k^k U.
\]
for all \( k \). \( \Box \)

In formula (10), we can express explicitly the dependence of the coefficients on the characteristic of \( U \) (see (9)). We get
\[
\sigma^k \tau^k U = (-1)^0 \Sigma_0^{m-l+2,k} (\tau \sigma)^k U + (-1)^1 \Sigma_1^{m-l+2,k-1} (\tau \sigma)^{k-1} U \\
+ (-1)^2 \Sigma_2^{m-l+2,k-2} \tau U + \ldots + (-1)^{k-1} \Sigma_{k-1}^{m-l+2,k-1} (\tau \sigma)^{k-1} U \\
+ (-1)^k \Sigma_k^{m-l+2,k} U.
\]
This dependence is essential in the following formula for the decomposition of the operator \( \sigma^k \tau^k \).

**Corollary 4.** For any tensor \( U \) of characteristic \((l, m)\), and all \( k = 0, 1, 2, \ldots, l \),
\[
\sigma^k \tau^k U = (-1)^0 \Sigma_0^{m-l+2,k} (\tau \sigma)^k U \\
+ (-1)^1 \Sigma_1^{m-l+2,k-1} (\tau \sigma)^{k-1} U + (-1)^2 \Sigma_2^{m-l+2,k-2} (\tau \sigma)^{k-2} U \\
+ \ldots + (-1)^{k-1} \Sigma_{k-1}^{m-l+2,k-1} (\tau \sigma)^{k-1} U \\
+ (-1)^k \Sigma_k^{m-l+2,k} (\tau \sigma)^k U.
\]

**Proof.** To prove (15), we apply (10) to tensors of the form \( \tau U \). Since the characteristic of these tensors is \((l-1, m+1)\), we have to use the polynomials \( \Sigma_q^{m-l+2,p} \) instead of \( \Sigma_q^{m-l,p} \). Thus,
\[
\sigma^k \tau^k \tau U = (-1)^0 \Sigma_0^{m-l+2,k-1} (\tau \sigma)^{k-1} \tau U \\
+ (-1)^1 \Sigma_1^{m-l+2,k-2} (\tau \sigma)^{k-2} \tau U + (-1)^2 \Sigma_2^{m-l+2,k-3} (\tau \sigma)^{k-3} \tau U \\
+ \ldots + (-1)^{k-1} \Sigma_{k-1}^{m-l+2,k-1} (\tau \sigma)^{k-1} \tau U \\
+ (-1)^k \Sigma_k^{m-l+2,k} (\tau \sigma)^k \tau U.
\]

Acting on (14) with \( \sigma \), we get
\[
\sigma^k \tau^k U = (-1)^0 \Sigma_0^{m-l+2,k} \sigma (\tau \sigma)^{k-1} \tau U \\
+ (-1)^1 \Sigma_1^{m-l+2,k} \sigma (\tau \sigma)^{k-2} \tau U + (-1)^2 \Sigma_2^{m-l+2,k} \sigma (\tau \sigma)^{k-3} \tau U \\
+ \ldots + (-1)^{k-1} \Sigma_{k-1}^{m-l+2,k} \sigma (\tau \sigma)^{k-1} \tau U \\
+ (-1)^k \Sigma_k^{m-l+2,k} \sigma (\tau \sigma)^k \tau U.
\]

Now (15) follows from the equality \( \sigma (\tau \sigma)^{k-1} \tau U = (\tau \sigma)^k U \). \( \Box \)

**Lemma 2.** (a) The operators \((\tau \sigma)^p\) and \(\sigma^k \tau^k\) satisfy
\[
(\tau \sigma)^p \sigma^k \tau^k U = \sigma^k \tau^k (\tau \sigma)^p U, \quad (\tau \sigma)^p \sigma^k \tau^k U = \sigma^k \tau^k (\tau \sigma)^p U.
\]
(b) \(\sigma^l \tau^l\) satisfies
\[
\sigma^l \tau^l \sigma^l \tau^l U = \frac{l! (m + l)!}{m!} \sigma^l \tau^l U,
\]
and
\[
(\tau \sigma)^k \sigma^l \tau^l U = \sigma^l \tau^l (\tau \sigma)^k U = m^k (l + 1)^k \sigma^l \tau^l U.
\]
Trace decompositions of tensor spaces

**Proof.** Formulas (16) are immediate consequences of (10) and (15), and (17) follows from (4). To derive (18) note that by Lemma 2, (a), \((\tau\sigma)^{l}\tau^{l}U = \sigma^{l}\tau^{l}(\tau\sigma)U\). But from (6),

\[
sym_{1,2}^{l}sym_{2,1}^{l}sym_{2,1}^{l}sym_{1,2}^{l}U
= \left( \frac{1}{l+1} + \frac{lm}{m(l+1)} \right) sym_{1,2}^{l}sym_{2,1}^{l}U = sym_{1,2}^{l}sym_{2,1}^{l}U.
\]

Thus, using (3) and (4), \((\tau\sigma)^{l}\tau^{l}U = \sigma^{l}\tau^{l}(\tau\sigma)U = m(l+1)\sigma^{l}\tau^{l}U\). Repeating this formula \(k\) times we get (18). \(\square\)

Now consider the operator \(\sigma^{k}\tau^{k} : Z^{(l,m)}E \rightarrow Z^{(l,m)}E\) for \(k = l\); note that this operator satisfies (4). We begin by proving an identity involving the symmetric polynomials \(\Sigma_{q}^{l}\) (7), (8).

**Theorem 11.** The polynomials \(\Sigma_{1}^{l}, \Sigma_{2}^{l}, \ldots, \Sigma_{l}^{l}\) satisfy

\[
(-1)^{0}\Sigma_{0}^{l}m^{l}(l+1)^{l} + (-1)^{1}\Sigma_{1}^{l}m^{l-1}(l+1)^{l-1} + (-1)^{2}\Sigma_{2}^{l}m^{l-2}(l+1)^{l-2}
+ \ldots + (-1)^{l-1}\Sigma_{l-1}^{l}m(l+1) + (-1)^{l}\Sigma_{l}^{l} = \frac{l!(m+l)!}{m!}.
\]

**Proof.** To derive (19), we write formula (10) for \(k = l\). We have

\[
\sigma^{l}\tau^{l}U = (-1)^{0}\Sigma_{0}^{l}(\tau\sigma)^{l}U + (-1)^{1}\Sigma_{1}^{l}(\tau\sigma)^{l-1}U + (-1)^{2}\Sigma_{2}^{l}(\tau\sigma)^{l-2}U
+ \ldots + (-1)^{l-1}\Sigma_{l-1}^{l}(\tau\sigma)U + (-1)^{l}\Sigma_{l}^{l}U.
\]

(19) now follows from (17), (18). \(\square\)

Let \(U\) be a given non-zero tensor of characteristic \((l,m)\), and let \(\lambda \in \mathbb{R}\) be a non-zero real number. We now study the equation

\[
X - \frac{1}{\lambda}\tau\sigma X = U
\]

for an unknown tensor \(X \in Z^{(l,m)}E\). We denote

\[
S_{\lambda}^{(l,m)}X = X - \frac{1}{\lambda}\tau\sigma X.
\]

\(S_{\lambda}^{(l,m)}\) is a linear operator on the tensor space \(Z^{(l,m)}E\). We want to find all \(\lambda\) such that there exists the inverse operator \(T_{\lambda}^{(l,m)}\) to \(S_{\lambda}^{(l,m)}\); for such \(\lambda\), the tensor

\[
X = T_{\lambda}^{(l,m)}U
\]
solves equation (20), for any \(U\).

To find \(T_{\lambda}^{(l,m)}\), denote

\[
U^{k} = U + \frac{1}{\lambda}(\tau\sigma)^{1}U + \frac{1}{\lambda^{2}}(\tau\sigma)^{2}U + \ldots + \frac{1}{\lambda^{k}}(\tau\sigma)^{k}U,
\]

and

\[
\mu(\lambda) = (-1)^{0}\lambda^{l}\Sigma_{0}^{l} + (-1)^{1}\lambda^{l-1}\Sigma_{1}^{l} + (-1)^{2}\lambda^{l-2}\Sigma_{2}^{l}
+ \ldots + (-1)^{l-2}\lambda^{2}\Sigma_{l-2}^{l} + (-1)^{l-1}\lambda^{l}\Sigma_{l-1}^{l} + (-1)^{l}\lambda^{0}\Sigma_{l}^{l}.
\]
Lemma 3. (a) \( \mu \) can be expressed as
\[
\mu(\lambda) = (\lambda - N_{l-1})(\lambda - N_{l-2})(\lambda - N_{l-3}) \ldots (\lambda - N_1)(\lambda - N_0).
\]
(b) At \( \lambda = \lambda_0 = (l + 1)m \),
\[
\mu(\lambda_0) = \frac{1}{m!}.
\]

Proof. (a) The proof is based on the properties (8) of the symmetric polynomials \( \Sigma_q \). We know that
\[
(-1)^0 \lambda^p \Sigma_0^p = (-1)^0 \lambda^p \Sigma_0^{p-1},
\]
\[
(-1)^1 \lambda^{p-1} \Sigma_1^p = (-1)^1 \lambda^{p-1} \Sigma_1^{p-1} + (-1)^1 \lambda^{p-1} \Sigma_0^{p-1} N_{p-1},
\]
\[
\ldots
\]
\[
(-1)^{p-1} \lambda^1 \Sigma_{p-1}^p = (-1)^{p-1} \lambda^1 \Sigma_{p-1}^{p-1} + (-1)^{p-1} \lambda^1 \Sigma_{p-2}^{p-1} N_{p-1},
\]
\[
(-1)^p \Sigma_p^p = (-1)^p \Sigma_{p-1}^{p-1} N_{p-1},
\]
i.e., after some calculation,
\[
(-1)^0 \lambda^p \Sigma_0^p + (-1)^1 \lambda^{p-1} \Sigma_1^p + (-1)^2 \lambda^{p-2} \Sigma_2^p + (-1)^3 \lambda^{p-3} \Sigma_3^p + \ldots + (-1)^{p-2} \lambda^2 \Sigma_{p-2}^p + (-1)^{p-1} \lambda \Sigma_{p-1}^p + (-1)^p \Sigma_p^p
\]
\[
= (\lambda - N_{p-1})(-1)^0 \lambda^{p-1} \Sigma_0^p + (-1)^1 \lambda^{p-2} \Sigma_1^p + (-1)^2 \lambda^{p-3} \Sigma_2^p + \ldots + (-1)^{p-2} \lambda \Sigma_{p-1}^p + (-1)^p \Sigma_p^p
\]
\[
= (\lambda - N_{p-1})(\lambda - N_{p-2})(\lambda - N_{p-3}) \ldots (\lambda - N_2)(\lambda - N_1)(\lambda - N_0).
\]
Since this equality holds for all \( p \), we have
\[
(-1)^0 \lambda^p \Sigma_0^p + (-1)^1 \lambda^{p-1} \Sigma_1^p + (-1)^2 \lambda^{p-2} \Sigma_2^p + (-1)^3 \lambda^{p-3} \Sigma_3^p + \ldots + (-1)^{p-2} \lambda^2 \Sigma_{p-2}^p + (-1)^{p-1} \lambda \Sigma_{p-1}^p + (-1)^p \Sigma_p^p
\]
\[
= (\lambda - N_{p-1})(\lambda - N_{p-2})(\lambda - N_{p-3}) \ldots (\lambda - N_2)(\lambda - N_1)(\lambda - N_0).
\]

For \( p = l \) we get (23).
(b) (24) follows from Theorem 2. \( \square \)

Now we are in a position to prove the following result.

Theorem 12. The operator \( S_{\lambda}^{(l,m)} \) is invertible if and only if
\[
\lambda \neq (k + 1)(m - l + k), \quad k = 0, 1, 2, \ldots, l - 1, l.
\]
In this case \( (\lambda - m(l + 1)) \mu(\lambda) \neq 0 \), and the inverse operator \( U \rightarrow T_{\lambda}^{(l,m)}U \) is given by
\[
T_{\lambda}^{(l,m)}U = \frac{\lambda}{(\lambda - m(l + 1)) \mu(\lambda)} \sigma^l \tau^l U
\]
\[
+ \frac{\lambda}{\mu(\lambda)} \left( (-1)^0 \Sigma_0^l \lambda^{l-1} U^{l-1} + (-1)^1 \Sigma_1^l \lambda^{l-2} U^{l-2} + (-1)^2 \Sigma_2^l \lambda^{l-3} U^{l-3} + \ldots + (-1)^{l-2} \Sigma_{l-2}^l \lambda^1 U^1 + (-1)^{l-1} \Sigma_{l-1}^l \lambda^0 U^0 \right).
\]
Proof. 1. Let $\lambda$ be such that there exists $T^{(l,m)}_\lambda$. Then for every $U$, there exists a unique tensor $X$ satisfying (20). We have for these $U$ and $X$

$$U = X - \frac{1}{\lambda} (\tau \sigma)^1 X,$$

$$\frac{1}{\lambda} (\tau \sigma)^1 U = \frac{1}{\lambda} (\tau \sigma)^1 X - \frac{1}{\lambda^2} (\tau \sigma)^2 X,$$

$$\frac{1}{\lambda^2} (\tau \sigma)^2 U = \frac{1}{\lambda^2} (\tau \sigma)^2 X - \frac{1}{\lambda^3} (\tau \sigma)^3 X,$$

$$\vdots$$

$$\frac{1}{\lambda^{k-1}} (\tau \sigma)^{k-1} U = \frac{1}{\lambda^{k-1}} (\tau \sigma)^{k-1} X - \frac{1}{\lambda^k} (\tau \sigma)^k X,$$

hence

$$U + \frac{1}{\lambda} (\tau \sigma)^1 U + \frac{1}{\lambda^2} (\tau \sigma)^2 U + \ldots + \frac{1}{\lambda^{k-1}} (\tau \sigma)^{k-1} U = X - \frac{1}{\lambda^k} (\tau \sigma)^k X,$$

i.e., for all $k \geq 1$,

$$\frac{1}{\lambda^k} (\tau \sigma)^k X = X - U^{k-1},$$

where

$$U^{k-1} = U + \frac{1}{\lambda} (\tau \sigma)^1 U + \frac{1}{\lambda^2} (\tau \sigma)^2 U + \ldots + \frac{1}{\lambda^{k-1}} (\tau \sigma)^{k-1} U$$

($U_0 = U$). On the other hand, by Theorem 1, (10),

$$\sigma^l \tau^l X = (-1)^0 \Sigma^l_0 (\tau \sigma)^l X + (-1)^1 \Sigma^l_1 (\tau \sigma)^l X + (-1)^2 \Sigma^l_2 (\tau \sigma)^l X + \ldots + (-1)^l \Sigma^l_l X.$$  

We now combine (26) and (27). We obtain

$$(\tau \sigma)^1 X = \lambda^1 (X - U^0),$$

$$(\tau \sigma)^2 X = \lambda^2 (X - U^1),$$

$$\vdots$$

$$(\tau \sigma)^l X = \lambda^l (X - U^{l-2}),$$

and, with $\mu(\lambda)$ defined by (22),

$$\sigma^l \tau^l X = \mu(\lambda) X - (-1)^0 \Sigma^l_0 \lambda^l U^{l-1} - (-1)^1 \Sigma^l_1 \lambda^l U^{l-2}$$

$$- (-1)^2 \Sigma^l_2 \lambda^{-2} U^{l-3} - \ldots - (-1)^{l-2} \Sigma^l_{l-2} \lambda^2 U^1 - (-1)^{l-1} \Sigma^l_{l-1} \lambda^l U^0.$$  

We find $\sigma^l \tau^l X$ from equation (20), and then compute $X$ from equation (28). From Lemma 2, (a) we know that $\tau \sigma \sigma^l \tau^l X = \sigma^l \tau^l \tau \sigma X = m(l + 1) \sigma^l \tau^l X$. Thus, we have from (20)

$$\sigma^l \tau^l U = \sigma^l \tau^l X - \frac{1}{\lambda} \sigma^l \tau^l \tau \sigma X$$

$$= \sigma^l \tau^l X - \frac{m(l + 1)}{\lambda} \sigma^l \tau^l X = \frac{\lambda - m(l + 1)}{\lambda} \sigma^l \tau^l X.$$
Substituting from (29) to (28) we obtain
\[
\frac{\lambda - m(l + 1)}{\lambda} \sigma^l \tau^l X = \frac{\lambda - m(l + 1)}{\lambda} \mu(\lambda) X
\]
\[
- \frac{\lambda - m(l + 1)}{\lambda} \left( (-1)^0 \sigma_0^l \lambda^l U^{l-1} + (-1)^1 \sigma_1^l \lambda^{l-1} U^{l-2} \right)
\]
\[
+ (-1)^2 \sigma_2^l \lambda^{l-2} U^{l-3} + \ldots + (-1)^l \sigma_l^l \lambda^0 U^{l-1}
\]
\[
= \sigma^l \tau^l U,
\]
that is,
\[
\frac{(\lambda - m(l + 1)) \mu(\lambda)}{\lambda} X = \sigma^l \tau^l U + \frac{\lambda - m(l + 1)}{\lambda} \left( (-1)^0 \sigma_0^l \lambda^l U^{l-1} \right)
\]
\[
+ (-1)^1 \sigma_1^l \lambda^{l-1} U^{l-2} + (-1)^2 \sigma_2^l \lambda^{l-2} U^{l-3} + \ldots + (-1)^l \sigma_l^l \lambda^0 U^{l-1}
\]
\[
+ (-1)^{l-1} \sigma_l^l \lambda^0 U^0.
\]
(30)

But by hypothesis, to the given \( U \) we have a unique tensor \( X \) satisfying (20); thus, \( \lambda \) must differ from the roots of the polynomial \((\lambda - m(l + 1)) \mu(\lambda)\).

From Lemma 3 we now see that (30) gives formula (25) of Theorem 3.

2. The converse can be proved by reversing our arguments. \( \square \)

**Remark 6.** If we denote the polynomial (22) by \( \mu^l \), then by Lemma 3, \( \mu^{l+1}(\lambda) = (\lambda - m(l + 1)) \mu^l(\lambda) \), i.e.,
\[
\mu^{l+1}(\lambda) = (\lambda - N_l)(\lambda - N_{l-1})(\lambda - N_{l-2})(\lambda - N_{l-3}) \ldots (\lambda - N_1)(\lambda - N_0).
\]

Thus, Theorem 3 says that the operator \( S_{\lambda}^{(l,m)} \) is invertible if and only if \( \lambda \) differs from the roots of the polynomial \( \mu^{l+1} \).

7. Multisymmetric-symmetric tensors

In Section 5, we obtained the trace decomposition equations for tensors \( A \in Z^l E \) by means of two operators, \( q \) and \( \text{tr} \). Recall that for a given tensor \( A \), the problem is to find traceless tensors \( X \in Z^{l-1}_q E \) and \( U \in Z^l_0 E \) such that
\[
(1) \quad A = q X + U.
\]

A solution \((X, U)\) can be easily found by means of the commutation formula for \( q \) and \( \text{tr} \); we have \( \text{tr} A = q \text{tr} X + \text{tr} U = X - q \text{tr} X + \text{tr} U = X \), because by hypothesis, \( \text{tr} X = 0 \) and \( \text{tr} U = 0 \); then \( U = A - q X \).

Now we wish to formulate the *complete trace decomposition equations* for tensor spaces \( A \in Z^l_{s_1, r_2, \ldots, r_p} E \), where \( 2 \leq p \leq s \). To this purpose we need analogues of the operators \( q \) and \( \text{tr} \), applicable to several multi-indices; we also need the elementary symmetrization operators introduced in Section 6, which could be applied to any pair of the multi-indices.

Thus, we define the operators \( q_\alpha \) and \( \text{tr}_\alpha \) by means of formulas (1), (2), Section 5, applied to the multi-index \( J_\alpha = (j_1 j_2 \ldots j_{r_\alpha}) \). For any two multi-indices \( J_\alpha = (j_1 j_2 \ldots j_{r_\alpha}) \) and \( J_\beta = (k_1 k_2 \ldots k_{r_\beta}) \), we define the elementary symmetrization operators \( \sigma_{\alpha, \beta} \) by means of formula (3), Section 6.
If, for example, $\alpha = 1$, $r_\alpha = l$, $\beta = 2$, $r_\beta = m$, and $U$ belongs to the tensor space $Z_s^{l,m} E$, $U = U_{j_1j_2\ldots j_i} k_{1k_2\ldots k_m} i_{i_1i_2\ldots i_s}$, we have

\[
q_1 U = C_{i_1} s_{j_1} U_{j_2j_3\ldots j_{i+1}} k_{1k_2\ldots k_m} i_{i_1i_2\ldots i_{i+1}} \text{alt} (i_1i_2\ldots i_{i+1}) \text{sym} (j_1j_2\ldots j_{i+1}),
\]

\[
q_2 U = C_{m} s_{j_1} U_{j_2j_3\ldots j_{i+1}} k_{1k_2\ldots k_m} i_{i_1i_2\ldots i_{i+1}} \text{alt} (i_1i_2\ldots i_{i+1}) \text{sym} (k_1k_2\ldots k_{m+1}),
\]

\[
\text{tr}_1 U = U_{j_1j_2\ldots j_{i-1}} k_{1k_2\ldots k_m} p_{i_1i_2\ldots i_{i-1}},
\]

\[
\text{tr}_2 U = U_{j_1j_2\ldots j_{i-1}} p_{k_1k_2\ldots k_{m-1}} i_{i_1i_2\ldots i_{i-1}},
\]

\[
\sigma_{1,2} U = (l+1) U_{j_2j_3\ldots j_{i-1}} j_1k_{1k_2\ldots k_{m-1}} i_{i_1i_2\ldots i_s} \text{sym} (j_1j_2\ldots j_{i-1}),
\]

\[
\sigma_{2,1} U = (m+1) U_{k_1k_2\ldots k_m} j_{1j_2\ldots j_{i-1}} k_{1k_2\ldots k_m} i_{i_1i_2\ldots i_s} \text{sym} (k_1k_2\ldots k_{m+1}).
\]

Using these definitions, we can prove the following two theorems. Denote

\[
\sigma = \sigma_{1,2} = (l+1) \text{sym}_{1,2}, \quad \tau = \sigma_{2,1} = (m+1) \text{sym}_{2,1}.
\]

**Theorem 13.** For $i = 1, 2$,

\[
q_1 q_1 U = 0, \quad \text{tr}_i \text{tr}_i U = 0, \quad U = q_1 \text{tr}_i U + \text{tr}_i q_1 U,
\]

\[
q_2 q_1 U = -\frac{n+l-s-1}{n+m-s-1} \frac{n+m-s}{n+l-s} q_1 q_2 U, \quad \text{tr}_1 \text{tr}_2 U = -\text{tr}_2 \text{tr}_1 U.
\]

**Proof.** See Section 5, Theorem 1. $\square$

**Theorem 14.** For any $U \in Z_s^{l,m} E$, $U = U_{j_1j_2\ldots j_i} k_{1k_2\ldots k_m}$,

\[
\text{tr}_2 q_1 U = \frac{1}{n+l-s} \sigma U - \frac{n+l-s+1}{n+l-s} q_1 \text{tr}_2 U,
\]

\[
\text{tr}_1 q_2 U = \frac{1}{n+m-s} \tau U - \frac{n+m-s+1}{n+m-s} q_2 \text{tr}_1 U,
\]

\[
\text{tr}_2 \tau U = \text{tr}_1 U + \tau \text{tr}_2 U,
\]

\[
\text{tr}_1 \sigma U = \text{tr}_2 U + \sigma \text{tr}_1 U,
\]

\[
\text{tr}_1 \tau U = \tau \text{tr}_1 U,
\]

\[
\text{tr}_2 \sigma U = \sigma \text{tr}_2 U,
\]

\[
\tau q_1 U = \frac{n+m-s}{n+l-s} q_2 U + \frac{n+l-1-s}{n+l-s} q_1 \tau U,
\]

\[
\sigma q_2 U = \frac{n+l-s}{n+m-s} q_1 U + \frac{n+m-1-s}{n+m-s} q_2 \sigma U,
\]

\[
\tau q_1 U = \frac{n+l+1-s}{n+l-s} q_1 \tau U,
\]

\[
\tau q_2 U = \frac{n+m+1-s}{n+m-s} q_2 \tau U,
\]

\[
q_2 \text{tr}_1 q_1 \text{tr}_2 U = \frac{1}{(n+m-s)(n+l-s+1)} \tau \sigma U - \frac{n+l-s}{(n+m-s)(n+l-s+1)} \tau q_2 U
\]
\[ + \frac{n + l - s}{n + l - s + 1} \text{tr}_1 q_1 (U - \text{tr}_2 q_2 U), \]
\[ \text{tr}_2 q_1 \text{tr}_1 q_2 U \]
\[ = \frac{1}{(n + l - 1 - s)(n + m - s)} (\sigma U - (n + m - s + 1)\sigma q_2 \text{tr}_1 U) \]
\[ - \frac{(n + l - s)}{(n + l - 1 - s)(n + m - s)} (q_1 \text{tr}_1 2 \tau U - (n + m - s + 1)q_1 \text{tr}_2 q_2 \text{tr}_1 U), \]
\[ \tau \sigma \text{tr}_1 \text{tr}_2 U = \text{tr}_1 \text{tr}_2 \sigma U, \]
\[ \sigma \text{tr}_1 \text{tr}_2 U = \text{tr}_1 \text{tr}_2 \sigma U, \]
\[ \tau \text{tr}_1 \text{tr}_2 U = \text{tr}_1 \text{tr}_2 \tau U. \]

**Proof.** The formulas can be obtained by a direct computation. □

From Theorem 1 we can now easily deduce the general structure of the trace decomposition equations for tensors \( A \in Z_s^{(r_1, \ldots, r_p)} E \), generalizing equation (1). If \( p = 2 \), then the trace decomposition equation for a tensor \( A \in Z_s^{(l, m)} E \) is

\[ (2) \quad A = q_2 q_1 X + q_1 U + q_2 V + Z, \]

where \( X, U, V, Z \) are unknown traceless tensors. If \( p = 3 \) and \( A \in Z_s^{(l, m, p)} E \), the trace decomposition equation is

\[ (3) \quad A = q_3 q_2 q_1 X + q_2 q_1 U_1 + q_3 q_1 U_2 + q_3 q_2 U_3 + q_1 V_1 + q_2 V_2 + q_3 V_3 + Z, \]

where \( X, U_1, U_2, U_3, V_1, V_2, V_3, Z \) are unknown traceless tensors. The general trace decomposition equation can be formulated in the same way; then Theorem 2 gives us the method for solving this equation.

8. The trace decomposition of tensors of characteristic \((l, m; s)\)

Let \( A \) be a tensor of characteristic \((l, m; s)\). The trace decomposition problem for \( A \) is the problem of solvability of the trace decomposition equation

\[ (1) \quad A = q_2 q_1 X + q_1 U + q_2 V + Z \]

for unknown traceless tensors \( X, U, V, Z \) (Section 7). Our main result in this section is the following explicit trace decomposition formula.

**Theorem 15.** For any tensor \( A \in Z_s^{(l, m)} E \), the tensors

\[ X = T_{\lambda_0}^{(l-1, m-1)} \text{tr}_1 \text{tr}_2 A, \]
\[ U = T_{\lambda_1}^{(l-1, m)} \text{tr}_1 (A - q_2 q_1 X) - \frac{1}{n - s + m} T_{\lambda_1}^{(l-1, m)} \tau \text{tr}_2 (A - q_2 q_1 X), \]
\[ V = T_{\lambda_1}^{(l, m-1)} \text{tr}_2 (A - q_2 q_1 X) - \frac{1}{n - s + l} T_{\lambda_1}^{(l, m-1)} \sigma \text{tr}_1 (A - q_2 q_1 X), \]
\[ Z = A - q_1 U - q_2 V - q_2 q_1 X \]

solve the trace decomposition equation (1).

**Proof.** In the following computations, we use formulas given in Theorem 1 and Theorem 2, Section 7.
Applying the operator $\text{tr}_2$ to (1) we have $\text{tr}_2 A = q_1 X - q_2 \text{tr}_2 q_1 X + \text{tr}_2 q_1 U + V$, and $\text{tr}_1 \text{tr}_2 A = X - \text{tr}_1 q_2 \text{tr}_2 q_1 X$. We compute the second term, $\text{tr}_1 q_2 \text{tr}_2 q_1 X$. Since $\text{char} X = (l-1, m-1; s-2)$ and $X$ is traceless, we have

$$
\text{tr}_2 q_1 X = \frac{1}{n+l-1-s+2} \sigma X - \frac{n+l-1-s+2+1}{n+l-1-s+2} q_1 \text{tr}_2 X = \frac{1}{n+l-s+1} \sigma X.
$$

Similarly, setting $V = \text{tr}_2 q_1 X$, we have $\text{char} V = (l, m-2; s-2)$. Thus,

$$
(2) \quad \text{tr}_1 q_2 \text{tr}_2 q_1 X = \frac{1}{n+m-s} \frac{1}{n+l-s+1} \tau \sigma X.
$$

Denoting $\lambda_0 = (n + m - s)(n + l - s + 1)$ we see that $X$ must satisfy the equation

$$
\text{tr}_1 \text{tr}_2 A = X - \frac{1}{\lambda_0} \tau \sigma X,
$$

or, in the notation of Section 6, (21),

$$
S^{(l-1,m-1)}_{\lambda_0} X = \text{tr}_1 \text{tr}_2 A.
$$

We know that for existence of a solution it is necessary and sufficient that $\lambda_0 \neq (k+1)(m-l+k)$ for every $k = 0, 1, 2, \ldots, l-2, l-1$. This condition is obviously satisfied: two solutions of the equation $(n + m - s)(n + l - s + 1) = (k+1)(m-l+k)$,

$k_1 = n - s + l$ and $k_2 = -n + s - m - 1$, satisfy $k_1 > l - 1$ and $k_2 = -n + s - m - 1 < 0$. Consequently,

$$
X = T^{(l-1,m-1)}_{\lambda_0} \text{tr}_1 \text{tr}_2 A.
$$

The trace decomposition equation (1) is thus reduced to the equation

$$
A - q_2 q_1 T^{(l-1,m-1)}_{\lambda_0} \text{tr}_1 \text{tr}_2 A = q_1 U + q_2 V + Z.
$$

Denote

$$
A' = A - q_2 q_1 T^{(l-1,m-1)}_{\lambda_0} \text{tr}_1 \text{tr}_2 A.
$$

Then we have

$$
\text{tr}_1 A' = \text{tr}_1 q_1 U + \text{tr}_1 q_2 V + \text{tr}_1 Z = U + \text{tr}_1 q_2 V,
$$

$$
\text{tr}_2 A' = \text{tr}_2 q_1 U + \text{tr}_2 q_2 V + \text{tr}_2 Z = \text{tr}_2 q_1 U + V.
$$

But $\text{char} U = (l-1, m, s-1)$ and $\text{char} V = (l, m-1, s-1)$, so that

$$
\text{tr}_2 q_1 U = \frac{1}{n+l-s} \sigma U, \quad \text{tr}_1 q_2 V = \frac{1}{n+m-s} \tau V,
$$

hence

$$
(4) \quad \text{tr}_1 A' = U + \frac{1}{n+m-s} \tau V, \quad \text{tr}_2 A' = \frac{1}{n+l-s} \sigma U + V.
$$

Applying symmetrizations, we have

$$
\sigma \text{tr}_1 A' = \sigma U + \frac{1}{n+m-s} \sigma \tau V, \quad \tau \text{tr}_2 A' = \frac{1}{n+l-s} \tau \sigma U + V.
$$
Then
\[
\frac{1}{n+l-s} \sigma \text{tr}_1 A' = \frac{1}{n+l-s} \sigma U + \frac{1}{n+l-s} \frac{1}{n+m-s} \sigma \tau V,
\]
\[
\frac{1}{n+m-s} \tau \text{tr}_2 A' = \frac{1}{n+m-s} \tau U + \frac{1}{n+m-s} \tau \sigma V.
\]

From (4)
\[
\frac{1}{n+l-s} \sigma \text{tr}_1 A' = \text{tr}_2 A' - V + \frac{1}{n+l-s} \frac{1}{n+m-s} \sigma \tau V,
\]
\[
\frac{1}{n+m-s} \tau \text{tr}_2 A' = \frac{1}{n+m-s} \tau U + \text{tr}_1 A' - U,
\]

Denoting \( \lambda_1 = (n-s+l)(n+s+m) \) we obtain the equations
\[
\text{tr}_2 A' - \frac{1}{n+l-s} \sigma \text{tr}_1 A' = V - \frac{1}{\lambda_1} \sigma \tau V,
\]
\[
\text{tr}_1 A' - \frac{1}{n+m-s} \tau \text{tr}_2 A' = U - \frac{1}{\lambda_1} \tau \sigma U.
\]

We can apply Theorem 3 of Section 6 to each of these equations.

We have to analyze solvability. Note that in (5), \( \text{char} V = (l, m - 1, s - 1) \); thus, for existence of a solution \( V \) it is necessary and sufficient that \( \lambda_1 \) be different from \( k = 0, 1, 2, \ldots, l-1, l \). Namely, we want to show that the solutions \( x \) of the equation
\[
(n-s+l)(n-s+m) = (x+1)(m-1-l+x)
\]
satisfy \( x \neq k \). Suppose that (6) has a (real) solution. Then the quadratic form
\[
x^2 + (m-l)x + m-1+l - (n-s+l)(n-s+m)
\]
has non-negative discriminant \( D = (m-l-2)^2 + 4(n-s+l)(n-s+m) \geq 0 \). \( D \) can be viewed as a non-negative quadratic form in \( n-s \),
\[
D = 4(n-s)^2 + 4(n-s)(m+l) + 4lm + (m-l-2)^2 \geq 0.
\]
Computing its discriminant, we have \( D' = 64(m-l-1) \) so (8) implies \( m-l-1 = 0 \).

In this case the double root of \( D \) is \( -(m+l)/2 \), and we have
\[
D = 4 \left( n-s + \frac{m+l}{2} \right)^2 = (2(n-s) + m+l)^2.
\]

The roots of the quadratic form (7) are given by \( x_1 = n-s+l, \ x_2 = -(n-s) - m \). Obviously \( x_{1,2} \neq 0, 1, 2, \ldots, l-1, l \) as desired.

An analogous result is true for the second equation (5). Writing (5) in the form
\[
S^{(l,m-1)}_{\lambda_1} V = \text{tr}_2 A' - \frac{1}{n+l-s} \sigma \text{tr}_1 A',
\]
\[
S^{(l,m-1)}_{\lambda_1} U = \text{tr}_1 A' - \frac{1}{n+m-s} \tau \text{tr}_2 A',
\]
we get the solution
\[
V = T^{(l,m-1)}_{\lambda_1}(\text{tr}_2 A' - \frac{1}{n-s+l} \sigma \text{tr}_1 A'),
\]
\[
U = T^{(l-1,m)}_{\lambda_1}(\text{tr}_1 A' - \frac{1}{n-s+m} \tau \text{tr}_2 A'),
\]
where \(A'\) is given by (3).

Finally, \(Z\) is determined from equation (1). Summarizing our results, we have
\[
Z = A - q_1 U - q_2 V - q_2 q_1 X,
\]
\[
X = T^{(l-1,m-1)}_{\lambda_0} \text{tr}_1 \text{tr}_2 A,
\]
\[
V = T^{(l,m-1)}_{\lambda_1}(\text{tr}_2 A' - \frac{1}{n-s+l} \sigma \text{tr}_1 A'),
\]
\[
U = T^{(l-1,m)}_{\lambda_1}(\text{tr}_1 A' - \frac{1}{n-s+m} \tau \text{tr}_2 A'),
\]
where \(A' = A - q_2 q_1 T^{(l-1,m-1)}_{\lambda_0} \text{tr}_1 \text{tr}_2 A\) is given by (3). This completes the proof. \(\Box\)

References


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