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Trace decompositions of tensor spaces

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Abstract. The trace decomposition theory of tensor spaces, based on duality, is presented. The trace decomposition equations for tensors, symmetric in some sets of superscripts, and antisymmetric in the subscripts, are derived by means of the trace operations and appropriate symmetrizations and antisymmetrizations; commutation relations for the corresponding linear operators are derived. Trace decompositions of various concrete tensor spaces are discussed.

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1. Introduction

The trace decomposition of tensors on an n-dimensional vector space E, endowed with a metric tensor, belongs to classical topics of the representation theory of the orthogonal group (see Weyl [20], Fulton and Harris [5]). According to this theory, one can decompose any tensor over E by means of the trace operation. Due to the presence of the metric tensor, the trace operations are defined not only in spaces of mixed tensors, but also in spaces of covariant, or contravariant tensors. Independent components in such a decomposition are traceless tensors, combined with the Kronecker δ-tensor. Further relevant information on this representation theory, as well as references, can be found e.g. in Hamermesh [8], Chapter 10, and in Welsh [19] (see also Gallot, Hulin, Lafontaine [6], Chapter III, K, and Naimark [17]).

However, this *metric* trace decomposition theory cannot be applied to vector spaces which do not carry a metric tensor.

On the other hand, the well-known *natural* (i.e., $GL_n(\mathbb{R})$-equivariant) trace operation, defined on the basis of *duality* of vector spaces, does not require any additional structure on the underlying vector space E. Conceptually, this operation differs from the metric one: It is defined *only* between the covariant and contravariant indices of a tensor, and is invariant with respect to the general linear group. In particular, the number of traces of a tensor of type (r, s) (i.e., rs), is in general smaller than the number of traces of a covariant tensor of type $(0, r + s)$ (i.e., $(1/2)(r + s)(r + s + 1)$).

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This paper is devoted to the trace decomposition theory of *mixed* tensors over a *real* vector space E, based on the concept of duality. In such a setting, the trace decomposition problem belongs to the theory of systems of linear equations, or the theory of natural projectors (Krupka [10], Krupka and Janyska [16]) rather than to the group representation theory. We give a survey of elementary concepts and those recent results, which have already been applied in differential geometry and the calculus of variations in fibered spaces (Krupka [10], [13], [14]). Our main goal is to present the trace decomposition equations for some types of tensors, and to discuss the solution methods for these equations.

In Section 2 we consider tensor spaces $T^r_s E$ of type (r, s) over a real, n-dimensional vector space E. We recall some standard definitions, and introduce the *trace mappings* and the *Kronecker tensors*. The *duality* between $T^r_s E$ and $T^s_r E$ is also studied. In Section 3 we present general *trace decomposition theorems*; the proofs are based on a modification of the classical Weyls method; changes are forced by the absence of the metric tensor on E.

In Section 4 we give an analysis of important special cases of the trace decomposition. These examples correspond with the torsion and curvature tensors studied in differential, and Finsler geometry (see e.g. Chern, Chen, Lam [3], Eisenhart [4], Gromoll, Klingenberg, Meyer [7]). It can be shown, in particular, that the well-known Weyl tensors, discovered on the basis of projective and conformal invariance (see Bokan [2], Eisenhart [4], Thomas [18], Weyl [21]), coincide with the traceless components of tensors of type$(1, 3)$, and $(2, 2)$ (Krupka [13], [14]). Our approach, represented by the trace decomposition formula, should be compared with the $O(n)$-irreducible decomposition of tensors of type $(1, 3)$ (Gallot, Hulin, Lafontaine, [6], Chapter III, K). Moreover, these examples clarify the non-uniqueness of the trace decomposition (Krupka [14], Kovár [9]).

Next sections are devoted to a difficult problem of finding the general trace decomposition formula. We discuss this problem for tensor spaces, describing underlying structures of global variational analysis, i.e., the spaces of differential forms on higher jet prolongations of fibered manifolds (Anderson [1], Krupka [11], [12], [15]). In Section 5 we give the complete trace decomposition formula for tensors, symmetric in contravariant, and antisymmetric in covariant indices (Krupka [12]). Section 6 is concerned with elementary symmetrization operators and their spectral properties, which allow us to extend the trace decomposition equations to multisymmetric-antisymmetric tensors. In Section 7 we find a suitable collection of operators, and commutation relations between them, giving us a method of solving these equations. Finally, we derive the complete trace decomposition formula for tensors, symmetric in two sets of contravariant indices, and antisymmetric in covariant indices (Section 8).

2. Tensor spaces

Throughout this paper, $\mathbb{R}$ is the field of real numbers, E is a real, n-dimensional vector space, and E^* is its dual. The *dual basis* to a basis $(e_1, e_2, \ldots, e_n)$ of E is denoted $(\mathrm{e}^1, e^2, \ldots, e^n)$. In a *fixed* basis e_i, a tensor $U \in T^r_s E$ is sometimes denoted by its components; we write $U = U^{j_1 j_2 \ldots j_r}{}_{i_1 i_2 \ldots i_s}$. The *Kronecker δ-symbols* δ^i_j and δ_{ij} are defined to be $0 \in \mathbb{R}$ if $i \neq j$ and $1 \in \mathbb{R}$ if $i = j$.

By a *tensor of type* (r, s) over E, where r, s are non-negative integers, we mean a *multilinear* mapping $U : E^* \times E^* \times \ldots \times E^* \times E \times E \times \ldots \times E \longrightarrow \mathbb{R}$ (r factors E^*, s factors E). A tensor of type $(r, 0)$ (respectively $(0, s)$) is also said to be *contravariant* (respectively *covariant*) of type r (respectively s). The vector space of tensors of type (r, s) over E is denoted by $T^r_s E$. $T^1_0 E$ (respectively $T^0_1 E$) can be canonically identified with E (respectively E^*).

To describe different tensor spaces, we use multi-indices $J = (j_1 j_2 \dots j_r)$, where $r \geq 1$ and $1 \leq j_1, j_2, \dots, j_r \leq n$; the lengths $|J| = r$ of different multi-indices do not necessarily coincide. We consider tensors $U = U^{J_1 J_2 \dots J_p}{}_{i_1 i_2 \dots i_s}$, *symmetric* in the superscripts entering each multi-index J_k, and antisymmetric in the subscripts; the *characteristic* of U, $\operatorname{char} U$, is defined to be the $(p+1)$-tuple $(r_1, r_2, \dots, r_p; s)$. The vector subspace of tensors of characteristic $(r_1, r_2, \dots, r_p; s)$ in $T_s^{r_1+r_2+\dots+r_p}E$ is denoted $\mathrm{Z}_s^{(r_1,r_2,\dots,r_p)}E$.

Tensors belonging to the tensor space $\mathrm{Z}_s^{(r_1,r_2,\dots,r_p)}E$ are sometimes called *p-multi-symmetric-antisymmetric*. If $p = 1$, we call these tensors *symmetric-antisymmetric*. If $s = 0$, we denote the corresponding tensor space by $\mathrm{Z}^{(r_1,r_2,\dots,r_p)}E$, and speak of *p-multisymmetric* tensors. For 2-multisymmetric-symmetric tensors, and for symmetric-antisymmetric tensors it is usually convenient to use the standard, explicit index notation.

If $U \in T_s^r E$ and $V \in T_q^p E$, the *tensor product* $U \otimes V \in T_{s+q}^{r+p} E$ is defined by

$$\begin{aligned}(U \otimes V)(\omega^1, \omega^2, \dots, \omega^{r+p}, \xi_1, \xi_2, \dots, \xi_{s+q}) \\ = U(\omega^1, \omega^2, \dots, \omega^r, \xi_1, \xi_2, \dots, \xi_s) V(\omega^{r+1}, \omega^{r+2}, \dots, \omega^{r+p}, \xi_{s+1}, \xi_{s+2}, \dots, \xi_{s+q})\end{aligned}$$

for all $\omega^\alpha \in E^*$, $\xi_\beta \in E$. The tensor $\delta = e_m \otimes e^m = \delta^i_j e_i \otimes e^j$ is the *Kronecker δ-tensor*. If e_i is a basis of E, then the tensors $e_{j_1} \otimes e_{j_2} \otimes \dots \otimes e_{j_r} \otimes e^{i_1} \otimes e^{i_2} \otimes \dots \otimes e^{i_s}$, $1 \leq i_1, i_2, \dots, i_s, j_1, j_2, \dots j_r \leq n$, form the *associated basis* of the vector space $T_s^r E$.

Let r and s be positive integers, let α and β be integers such that $1 \leq \alpha \leq r$, $1 \leq \beta \leq s$, and let e_i be a basis of E. Let $U \in T_s^r E$, $U = U^{j_1 j_2 \dots j_r}{}_{i_1 i_2 \dots i_s}$, be a tensor of type (r, s). We define a tensor $\operatorname{tr}^\alpha_\beta U \in T_{s-1}^{r-1} E$ by

$$\operatorname{tr}^\alpha_\beta U = V^{j_1 j_2 \dots j_{r-1}}{}_{i_1 i_2 \dots i_{s-1}} e_{j_1} \otimes e_{j_2} \otimes \dots \otimes e_{j_{r-1}} \otimes e^{i_1} \otimes e^{i_2} \otimes \dots \otimes e^{i_{s-1}},$$

where

$$V^{j_1 j_2 \dots j_{r-1}}{}_{i_1 i_2 \dots i_{s-1}} = U^{j_1 j_2 \dots j_{\alpha-1} p j_\alpha j_{\alpha+1} \dots j_{r-1}}{}_{i_1 i_2 \dots i_{\beta-1} p i_\beta i_{\beta+1} \dots i_{s-1}}.$$

This tensor does not depend on the choice of the basis e_i. $\operatorname{tr}^\alpha_\beta U$ is the (α,β)-*trace* of U; the mapping $\operatorname{tr}^\alpha_\beta : T_s^r E \longrightarrow T_{s-1}^{r-1} E$ is the (α,β)-*trace mapping*. A tensor $U \in T_s^r E$ is called *traceless*, if $\operatorname{tr}^\alpha_\beta U = 0$ for all α, β.

Let $V \in T_{s-1}^{r-1} E$. We define a tensor $\iota^\alpha_\beta V \in T_s^r E$ by

$$\begin{aligned}\iota^\alpha_\beta V &= V^{j_1 j_2 \dots j_{r-1}}{}_{i_1 i_2 \dots i_{s-1}} e_{j_1} \otimes e_{j_2} \otimes \dots \otimes e_{j_{\alpha-1}} \otimes e_p \otimes e_{j_\alpha} \otimes \dots \otimes e_{j_{r-1}} \\ &\otimes e^{i_1} \otimes e^{i_2} \otimes \dots \otimes e^{i_{\beta-1}} \otimes e^p \otimes e^{i_\beta} \otimes \dots \otimes e^{i_{s-1}} \\ &= V^{j_1 j_2 \dots j_{\alpha-1} j_{\alpha+1} \dots j_r}{}_{i_1 i_2 \dots i_{\beta-1} i_{\beta+1} \dots i_s} \delta^{j_\alpha}_{i_\beta} e_{j_1} \otimes e_{j_2} \otimes \dots \otimes e_{j_r} \otimes e^{i_1} \otimes e^{i_2} \otimes \dots \otimes e^{i_s}.\end{aligned}$$

This tensor is independent of the choice of e_i. The mapping $\iota^\alpha_\beta : T_{s-1}^{r-1} E \longrightarrow T_s^r E$ is called the (α,β)-*canonical injection*.

A tensor $U \in T_s^r E$ belonging to the vector subspace generated by the subspaces $\iota^\alpha_\beta(T_{s-1}^{r-1} E) \subset T_s^r E$, where $1 \leq \alpha \leq r$, $1 \leq \beta \leq s$, is said to be a *Kronecker* (or *δ-generated*) tensor. It follows from the definition that the components of a Kronecker tensor are expressible in the form of linear combinations of terms, containing the Kronecker δ-symbol.

The *canonical pairing* of the vector spaces $T_s^r E$ and $T_r^s E$ is the bilinear mapping $T_s^r E \times T_r^s E \ni (U, V) \longrightarrow \langle U, V \rangle \in \mathbb{R}$ defined as follows: If in a basis of E,

$$U = U^{j_1 j_2 \dots j_r}{}_{i_1 i_2 \dots i_s}, \qquad V = V^{k_1 k_2 \dots k_s}{}_{l_1 l_2 \dots l_r},$$

then

$$\langle U, V \rangle = U^{j_1 j_2 \dots j_r}{}_{i_1 i_2 \dots i_s} V^{i_1 i_2 \dots i_s}{}_{j_1 j_2 \dots j_r}. \tag{1}$$

It is easily seen that this expression is independent of the choice of e_i.

Tensors $U \in T^r_s E$ and $V \in T^s_r E$ are said to be *orthogonal*, if $\langle U, V\rangle = 0$. The *orthogonal subspace* to a set $P \subset T^r_s E$ is the vector subspace Q of all vectors $V \in T^s_r E$ satisfying $\langle U, V\rangle = 0$ for every $U \in P$.

The structure of tensor spaces allows us to induce isomorphisms between tensor spaces $T^r_s E$ and $T^s_r E$ by means of isomorphisms of E and E^*. We construct these induced isomorphisms from symmetric regular bilinear forms on E. Let us recall basic definitions. Let $g \in T^0_2 E$. We say that g is *symmetric*, if $g(\xi, \zeta) = g(\zeta, \xi)$ for all $\xi, \zeta \in E$. For every $\zeta \in E$, define a linear form $g_\zeta \in E^*$ as the mapping $E \ni \xi \longrightarrow g_\zeta(\xi) = g(\xi, \zeta) \in \mathbb{R}$. We say that g is *regular*, if the mapping $\zeta \longrightarrow g^1_0(\zeta)$ of E into E^* is a linear isomorphism; in this case the *inverse* linear isomorphism of E^* into E is denoted by g^0_1.

g^1_0 is immediately extended to a linear isomorphism $g^r_s : T^r_s E \longrightarrow T^s_r E$. We set for every $U \in T^r_s E$, and all $\omega^1, \omega^2, \dots, \omega^r \in E^*$, $\xi_1, \xi_2, \dots, \xi_r \in E$,

$$\begin{aligned} &g^r_s U(\omega^1, \omega^2, \dots, \omega^s, \xi_1, \xi_2, \dots, \xi_r) \\ &\quad = U(g^1_0\xi_1, g^1_0\xi_2, \dots, g^1_0\xi_r, g^0_1\omega^1, g^0_1\omega^2, \dots, g^0_1\omega^s). \end{aligned}$$

g^r_s is said to be *generated* by g. Note that while U contains r covector arguments and s vector arguments, $g^s_r U$ depends on s covectors and r vectors.

Let in a basis e_i of E, $g = g_{ij} e^i \otimes e^j$. Since the matrix g_{ij} is regular, the *inverse* matrix g^{ij} satisfies $g_{ij} g^{jk} = \delta^k_i$. Let $\xi, \zeta \in E$, $\omega, \eta \in E^*$. Write $\xi = \xi^p e_p$, $\zeta = \zeta^p e_p$, $\omega = \omega_i e^i$, and $\eta = \eta_i e^i$; then $(g^1_0 \xi)(\zeta) = g_{ij} \xi^i e^j(\zeta)$ and $(g^0_1 \omega)(\eta) = g^{ij} \omega_i e_j(\eta)$. Consequently, $g^0_1 \omega = g^{ij} \omega_i e_j$, $g^1_0 \xi = g_{ij} \xi^i e^j$, and

$$\begin{aligned} g^r_s U &= g^{k_1 l_1} g^{k_2 l_2} \dots g^{k_s l_s} g_{i_1 j_1} g_{i_2 j_2} \dots g_{i_r j_r} U^{j_1 j_2 \dots j_r}{}_{l_1 l_2 \dots l_s} \\ &\quad \cdot e_{k_1} \otimes e_{k_2} \otimes \dots \otimes e_{k_s} \otimes e^{i_1} \otimes e^{i_2} \otimes \dots \otimes e^{i_r}, \end{aligned}$$

where

$$U(e^{j_1}, e^{j_2}, \dots, e^{j_r}, e_{l_1}, e_{l_2}, \dots, e_{l_s}) = U^{j_1 j_2 \dots j_r}{}_{l_1 l_2 \dots l_s}.$$

In the following lemmas we collect elementary properties of bilinear forms on tensor spaces. The proofs are immediate consequences of definitions.

Lemma 1. (a)*For any regular, symmetric bilinear form g on E*

$$\operatorname{tr}^\alpha_\beta \circ g^r_s = g^{r-1}_{s-1} \circ \operatorname{tr}^\beta_\alpha, \qquad g^r_s \circ \iota^\alpha_\beta = \iota^\beta_\alpha \circ g^{r-1}_{s-1}.$$

(b) *Let $U \in T^r_s E$. Then $\operatorname{tr}^\beta_\alpha U = 0$ if and only if $\operatorname{tr}^\alpha_\beta g^r_s U = 0$. In particular, U is traceless if and only if $g^r_s U$ is traceless.*

(c) *$U \in T^r_s E$ is a Kronecker tensor if and only if $g^r_s U$ is a Kronecker tensor.*

Every *scalar product* g on E induces a scalar product on $T^r_s E$. Namely, using the same notation as for the scalar product on E, we set for all $U, V \in T^r_s E$, $g(U, V) = \langle U, g^r_s V\rangle$. In a basis

(2) $$g(U, V) = g_{j_1 k_1} g_{j_2 k_2} \dots g_{j_r k_r} g^{i_1 l_1} g^{i_2 l_2} \dots g^{i_s l_s} U^{j_1 j_2 \dots j_r}{}_{i_1 i_2 \dots i_s} V^{k_1 k_2 \dots k_r}{}_{l_1 l_2 \dots l_s}.$$

Lemma 2. *Formula* (2) *defines a scalar product on $T^r_s E$.*

If g is a scalar product on E, then formula (2) allows us to represent the canonical pairing $(U, V) \longrightarrow \langle U, V\rangle$ in terms of g. Writing $V = g^r_s g^s_r V$ we get

(3) $$\langle U, V\rangle = \langle U, g^r_s g^s_r V\rangle = g(U, g^s_r V).$$

This expression is independent of the choice of g. In particular, formula (3) shows that $U \in T^r_s E$ and $V \in T^s_r E$ are *orthogonal* if and only if U and $g^s_r V$ are orthogonal with respect to the scalar product (2).

Let σ (respectively τ) be a permutation of the set $\{1, 2, \dots, r\}$ (respectively $\{1, 2, \dots, s\}$), and let $\xi_1, \xi_2, \dots, \xi_s \in E$ (respectively $\omega^1, \omega^2, \dots, \omega^r \in E^*$) be any vectors (respectively covectors). For every $U \in T^r_s E$ we define a tensor $(\sigma, \tau)U \in T^r_s E$ by

$$\begin{aligned}&((\sigma,\tau)U)(\omega^1,\omega^2,\dots,\omega^r,\xi_1,\xi_2,\dots,\xi_s)\\&\quad= U(\omega^{\sigma(1)},\omega^{\sigma(2)},\dots,\omega^{\sigma(r)},\xi_{\tau(1)},\xi_{\tau(2)},\dots,\xi_{\tau(s)}).\end{aligned}$$

The linear mapping $T^r_s E \ni U \longrightarrow (\sigma, \tau)U \in T^r_s E$ is called a *permutation* of $T^r_s E$. In a basis, $(\sigma, \tau)U$ has an expression

$$(\sigma,\tau)U = U^{i_{\sigma(1)}i_{\sigma(2)}\dots i_{\sigma(r)}}{}_{k_{\tau(1)}k_{\tau(2)}\dots k_{\tau(s)}} e_{i_1}\otimes e_{i_2}\otimes\dots\otimes e_{i_r}\otimes e^{k_1}\otimes e^{k_2}\otimes\dots\otimes e^{k_s}.$$

Lemma 3. *Let g be a scalar product on E, and let (σ, τ) be a permutation. Then*

$$g((\sigma,\tau)U, V) = g(U, (\sigma^{-1},\tau^{-1})V). \tag{4}$$

Corollary 1. (a) *For every scalar product g on E, $(\sigma, \tau)$is an orthogonal transformation of the vector space $T^r_s E$, i.e., $g((\sigma, \tau)U, (\sigma, \tau)V)) = g(U, V)$.*

(b) *If σ and τ are transpositions, then (σ, τ) is a symmetric transformation.*

3. The trace decomposition

In this section we suppose we have a fixed basis of E. Recall that a tensor $U \in T^r_s E$ is said to be *traceless*, if its traces are all zero, i.e.,

$$\begin{aligned}&U^{mi_2i_3\dots i_r}{}_{mk_2k_3\dots k_s}=0,\ U^{mi_2i_3\dots i_r}{}_{k_1mk_3k_4\dots k_s}=0,\dots,\\&\quad U^{mi_2i_3\dots i_r}{}_{k_1k_2\dots k_{s-1}m}=0,\\&U^{i_1mi_3i_4\dots i_r}{}_{mk_2k_3\dots k_s}=0,\ U^{i_1mi_3i_4\dots i_r}{}_{k_1mk_3k_4\dots k_s}=0,\dots,\\&\quad U^{i_1mi_3i_4\dots i_r}{}_{k_1k_2\dots k_{s-1}m}=0,\\&\quad\dots\\&U^{i_1i_2\dots i_{r-1}m}{}_{mk_2k_3\dots k_s}=0,\ U^{i_1i_2\dots i_{r-1}m}{}_{k_1mk_3k_4\dots k_s}=0,\dots,\\&\quad U^{i_1i_2\dots i_{r-1}m}{}_{k_1k_2\dots k_{s-1}m}=0.\end{aligned}\tag{1}$$

A tensor $V \in T^r_s E$ is said to be a *Kronecker tensor*, if there exist tensors $V^{(p)}_{(q)} \in T^{r-1}_{s-1}E$, where $1 \le p \le r$, $1 \le q \le s$, such that V can be expressed in the form

$$\begin{aligned}V^{k_1k_2\dots k_r}{}_{l_1l_2\dots l_s} &= \delta^{k_1}_{l_1}V^{(1)k_2k_3\dots k_r}_{(1)}{}_{l_2l_3\dots l_s} + \delta^{k_1}_{l_2}V^{(1)k_2k_3\dots k_r}_{(2)}{}_{l_1l_3\dots l_s}\\&+\dots+\delta^{k_1}_{l_s}V^{(1)k_2k_3\dots k_r}_{(s)}{}_{l_1l_2\dots l_{s-1}} + \delta^{k_2}_{l_1}V^{(2)k_1k_3\dots k_r}_{(1)}{}_{l_2l_3\dots l_s}\\&+\delta^{k_2}_{l_2}V^{(2)k_1k_3\dots k_r}_{(2)}{}_{l_1l_3\dots l_s}+\dots+\delta^{k_2}_{l_s}V^{(2)k_1k_3\dots k_r}_{(s)}{}_{l_1l_2\dots l_{s-1}}\\&+\dots+\delta^{k_r}_{l_1}V^{(r)k_1k_2\dots k_{r-1}}_{(1)}{}_{l_2l_3\dots l_s}+\delta^{k_r}_{l_2}V^{(r)k_1k_2\dots k_{r-1}}_{(2)}{}_{l_1l_3\dots l_s}\\&+\dots+\delta^{i_r}_{l_s}V^{(r)k_1k_2\dots k_{r-1}}_{(s)}{}_{l_1l_2\dots l_{s-1}}.\end{aligned}\tag{2}$$

The following result is the *trace decomposition theorem.*

Theorem 1. *The vector space $T^r_s E$ is the direct sum of its subspaces of traceless and Kronecker tensors.*

Proof. Existence and uniqueness can be proved by means of a scalar product on $T^r_s E$ (Section 2, Lemma 2). In the scalar product, the subspaces of traceless and Kronecker tensors become orthogonal. □

Theorem 1 can be rephrased more explicitly as follows.

Corollary 2. *Let n, r, s be positive integers, and let $W \in T^r_s E$, $W = W^{i_1 i_2 \dots i_r}{}_{k_1 k_2 \dots k_s}$. There exist a unique traceless tensor $U \in T^r_s E$, and tensors $V^{(p)}_{(q)} \in T^{r-1}_{s-1} E$, where $1 \le p \le r$, $1 \le q \le s$, such that*

$$
\begin{aligned}
W^{i_1 i_2 \dots i_r}{}_{l_1 l_2 \dots l_s} &= U^{i_1 i_2 \dots i_r}{}_{l_1 l_2 \dots l_s} \\
&+ \delta^{i_1}_{l_1} V^{(1) i_2 i_3 \dots i_r}_{(1)}{}_{l_2 l_3 \dots l_s} + \delta^{i_1}_{l_2} V^{(1) i_2 i_3 \dots i_r}_{(2)}{}_{l_1 l_3 \dots l_s} + \dots + \delta^{i_1}_{l_s} V^{(1) i_2 i_3 \dots i_r}_{(s)}{}_{l_1 l_2 \dots l_{s-1}} \\
&+ \delta^{i_2}_{l_1} V^{(2) i_1 i_3 \dots i_r}_{(1)}{}_{l_2 l_3 \dots l_s} + \delta^{i_2}_{l_2} V^{(2) i_1 i_3 \dots i_r}_{(2)}{}_{l_1 l_3 \dots l_s} + \dots + \delta^{i_2}_{l_s} V^{(2) i_1 i_3 \dots i_r}_{(s)}{}_{l_1 l_2 \dots l_{s-1}} \\
&+ \delta^{i_3}_{l_1} V^{(3) i_1 i_2 i_4 \dots i_r}_{(1)}{}_{l_2 l_3 \dots l_s} + \delta^{i_3}_{l_2} V^{(3) i_1 i_2 i_4 \dots i_r}_{(2)}{}_{l_1 l_3 \dots l_s} \\
&+ \dots + \delta^{i_3}_{l_s} V^{(3) i_1 i_2 i_4 \dots i_r}_{(s)}{}_{l_1 l_2 \dots l_{s-1}} \quad + \dots + \delta^{i_r}_{l_1} V^{(r) i_1 i_2 \dots i_{r-1}}_{(1)}{}_{l_2 l_3 \dots l_s} \\
&+ \delta^{i_r}_{l_2} V^{(r) i_1 i_2 \dots i_{r-1}}_{(2)}{}_{l_1 l_3 \dots l_s} + \dots + \delta^{i_r}_{l_s} V^{(r) i_1 i_2 \dots i_{r-1}}_{(s)}{}_{l_1 l_2 \dots l_{s-1}}.
\end{aligned} \tag{3}
$$

Formula (3) is the *trace decomposition formula* for the tensor W. (3) can also be viewed as the *trace decomposition equations* for the unknown tensors U and $V^{(p)}_{(q)}$.

Remark 1. The direct sum in Theorem 1, or Corollary 1, is independent of the choice of the auxiliary metric tensor g.

Remark 2. Both the *traceless component* U and the complementary *Kronecker component* in (3) are unique. However, this fact does not imply, in general, the uniqueness of the tensors $V^{(p)}_{(q)}$ (see Section 4).

We can apply the trace decomposition formula (3) to each of the tensors $V^{(p)}_{(q)} \in T^{r-1}_{s-1} E$. Repeating this step as many times as possible, we get the *complete trace decomposition* of U. To formulate the result more precisely, it is convenient to use explicit expressions. We need a specific summation convention allowing us to generalize formula (3). Suppose for example that $r \le s$. Let p and l be fixed integers such that $1 \le l \le p$. By an *l-partition* of the set $\{1, 2, \dots, l, l+1, \dots, p\}$ we mean two *ordered* subsets $\lambda_{(l,p)} = \{\lambda_1, \lambda_2, \dots, \lambda_l\}$ and $\Lambda_{(l,p)} = \{\Lambda_{l+1}, \Lambda_{l+2}, \dots, \Lambda_p\}$ of the set $\{1, 2, \dots, p\}$ such that $\lambda_{(l,p)} \bigcap \Lambda_{(l,p)} = \emptyset$, $\lambda_{(l,p)} \bigcup \Lambda_{(l,p)} = \{1, 2, \dots, l, l+1, \dots, p\}$. Thus, we have the inequalities $\lambda_1 < \lambda_2 < \dots < \lambda_l$, and $\Lambda_{l+1} < \Lambda_{l+2} < \dots < \Lambda_p$, and it is clear that to define an l-partition it is sufficient to choose one of the sets $\lambda_{(l,p)}$, $\Lambda_{(l,p)}$.

Let $W \in T^r_s E$, $W = W^{i_1 i_2 \dots i_r}{}_{k_1 k_2 \dots k_s}$, and let l be an integer such that $1 \le l \le r$. W is said to be *$\delta^{(l)}$-generated*, if it has an expression of the form

$$
\begin{aligned}
W^{i_1 i_2 \dots i_r}{}_{k_1 k_2 \dots k_s} &= \sum_{\lambda_{(l,r)}} \sum_{\xi_{(l,s)}} \sum_{\sigma \in \Sigma_l} \delta^{i_{\lambda_1}}_{k_{\varphi_{\sigma(1)}}} \delta^{i_{\lambda_2}}_{k_{\varphi_{\sigma(2)}}} \dots \delta^{i_{\lambda_l}}_{k_{\varphi_{\sigma(l)}}} \\
&\cdot V^{(\lambda_1, \lambda_2, \dots, \lambda_l)}_{i_{(\xi_{\sigma(1)}, \xi_{\sigma(2)}, \dots, \xi_{\sigma(l)})}}{}^{i_{\Lambda_{l+1}} {}^{\Lambda_{l+2}} \dots {}^{i_{\Lambda_r}}}{}_{k_{\vartheta_{l+1}} k_{\vartheta_{l+2}} \dots k_{\vartheta_r} k_{\vartheta_{r+1}} \dots k_{\vartheta_s}},
\end{aligned} \tag{4}
$$

where the tensor

$$V^{(\lambda_1,\lambda_2,\dots,\lambda_l)}_{(\varphi_{\sigma(1)},\varphi_{\sigma(2)},\dots,\varphi_{\sigma(l)})} = V^{(\lambda_1,\lambda_2,\dots,\lambda_l)}_{(\varphi_{\sigma(1)},\varphi_{\sigma(2)},\dots,\varphi_{\sigma(l)})}{}^{i_1 i_2\dots i_{r-l}}{}_{k_1 k_2\dots k_{s-l}} \tag{5}$$

belongs to $T^{r-l}_{s-l}E$. Each term in formula (4) is uniquely determined by two sets $\lambda_{(l,r)} \subset \{1,2,\dots,l,l+1,\dots,r\}$ and $\xi_{(l,s)} \subset \{1,2,\dots,l,l+1,\dots,s\}$, and by an element σ of the permutation group Σ_l of the set $\{1,2,\dots,l\}$. Note that (4) is a formal expression for the sum of terms in which all possible products of l Kronecker δ-tensors appear, with superscripts from the set $\{i_1,i_2,\dots,i_r\}$, and subscripts from the set $\{k_1,k_2,\dots,k_s\}$.

Obviously, $\delta^{(1)}$-generated tensors coincide with the Kronecker (i.e. δ-generated) tensors, which have already been defined. A $\delta^{(l)}$-generated tensor $W \in T^r_s E$ is called *l-primitive*, if it has a representation (4) in which all the tensors (5) are traceless; then the tensors (4) are called *$\delta^{(l)}$-components* of W. The vector subspace of $T^r_s E$ formed by l-primitive tensors is called *l-primitive*.

Remark 3. The structure of $\delta^{(l)}$-generated tensors can be described by means of the permutation operators. Clearly, every summand in (4) is of the form $U = (\sigma,\tau)(\delta\otimes\delta\otimes\dots\otimes\delta\otimes V)$ (l factors the Kronecker δ-tensors), where σ (respectively τ) is a permutation of the set $\{1,2,\dots,r\}$ (respectively $\{1,2,\dots,s\}$), and $V \in T^{r-l}_{s-l}E$.

The following is the *complete trace decomposition theorem.*

Theorem 2. *The vector space $T^r_s E$ is the direct sum of l-primitive subspaces.*

Proof. From Theorem 1 it follows that every tensor $W \in T^r_s E$ can be expressed as the sum of l-primitive tensors, where $l = 0,1,2,\dots,\min(r,s)$. We want to show that if W is l-primitive and m-primitive, and $l \neq m$, then $W = \{0\}$.

Let $U \in T^r_s E$ (respectively $V \in T^r_s E$) be l-primitive (respectively m-primitive), and suppose for instance that $m < l$. We show that in a scalar product g on E, $g(U,V) = 0$. It is sufficient to prove this equality for tensors of the form

$$\begin{aligned}
U^{i_1 i_2\dots i_r}{}_{k_1 k_2\dots k_s} &= \delta^{i_{\lambda_1}}_{k_{\xi_1}}\delta^{i_{\lambda_2}}_{k_{\xi_2}}\dots\delta^{i_{\lambda_l}}_{k_{\xi_l}} \\
&\cdot U^{(\lambda_1,\lambda_2,\dots,\lambda_l) i_{\Lambda_{l+1}} i_{\Lambda_{l+2}}\dots i_{\Lambda_r}}_{(\xi_1,\xi_2,\dots,\xi_l)}{}_{k_{\vartheta_{l+1}} k_{\vartheta_{l+2}}\dots k_{\vartheta_r} k_{\vartheta_{r+1}}\dots k_{\vartheta_s}}, \\
V^{j_1 j_2\dots j_r}{}_{l_1 l_2\dots l_s} &= \delta^{j_{\theta_1}}_{l_{\pi_1}}\delta^{j_{\theta_2}}_{l_{\pi_2}}\dots\delta^{j_{\theta_m}}_{l_{\pi_m}} \\
&\cdot V^{(\theta_1,\theta_2,\dots,\theta_m) j_{\Theta_{m+1}} j_{\Theta_{m+2}}\dots j_{\Theta_r}}_{(\pi_1,\pi_2,\dots,\pi_m)}{}_{l_{\Pi_{m+1}} l_{\Pi_{m+2}}\dots l_{\Pi_r} l_{\Pi_{r+1}}\dots l_{\Pi_s}},
\end{aligned}$$

determined by some partitions

$$\begin{aligned}
&\lambda_{(l,r)} = \{\lambda_1,\lambda_2,\dots,\lambda_l\}, \quad \Lambda_{(l,r)} = \{\Lambda_{l+1},\Lambda_{l+2},\dots,\Lambda_r\}, \\
&\xi_{(l,s)} = \{\xi_1,\xi_2,\dots,\xi_l\}, \quad \vartheta_{(l,s)} = \{\Xi_{l+1},\Xi_{l+2},\dots,\Xi_r,\Xi_{r+1},\dots,\Xi_s\},
\end{aligned}$$

and

$$\begin{aligned}
&\theta_{(m,r)} = \{\theta_1,\theta_2,\dots,\theta_m\}, \quad \Theta_{(m,r)} = \{\Theta_{m+1},\Theta_{m+2},\dots,\Theta_r\}, \\
&\pi_{(m,s)} = \{\pi_1,\pi_2,\dots,\pi_m\}, \quad \Pi_{(m,s)} = \{\Pi_{m+1},\Pi_{m+2},\dots,\Pi_r,\Pi_{r+1},\dots,\Pi_s\},
\end{aligned}$$

and by some permutations $\sigma, \nu \in \Sigma_l$. We have by Section 2, (2)

$$g(U,V) = g_{j_1 i_1} g_{j_2 i_2} \dots g_{j_r i_r} g^{k_1 l_1} g^{k_2 l_2} \dots g^{k_s l_s}$$
$$\cdot \delta^{i_{\lambda_1}}_{k_{\zeta_1}} \delta^{i_{\lambda_2}}_{k_{\zeta_2}} \dots \delta^{i_{\lambda_l}}_{k_{\zeta_l}} U^{(\lambda_1,\lambda_2,\dots,\lambda_l) i_{\Lambda_{l+1}} i_{\Lambda_{l+2}} \dots i_{\Lambda_r}}_{(\xi_1,\xi_2,\dots,\xi_l) \qquad\qquad k_{\Xi_{l+1}} k_{\Xi_{l+2}} \dots k_{\Xi_r} k_{\Xi_{r+1}} \dots k_{\Xi_s}} \tag{6}$$
$$\cdot \delta^{j_{\theta_1}}_{l_{\pi_1}} \delta^{j_{\theta_2}}_{l_{\pi_2}} \dots \delta^{j_{\theta_m}}_{l_{\pi_m}} V^{(\theta_1,\theta_2,\dots,\theta_m) j_{\Theta_{m+1}} j_{\Theta_{m+2}} \dots j_{\Theta_r}}_{(\pi_1,\pi_2,\dots,\pi_m) \qquad\qquad l_{\Pi_{m+1}} l_{\Pi_{m+2}} \dots l_{\Pi_r} l_{\Pi_{r+1}} \dots l_{\Pi_s}}.$$

Note that (6) includes contractions in $i_{\lambda_1}, i_{\lambda_2}, \dots, i_{\lambda_l}$, and in $l_{\pi_1}, l_{\pi_2}, \dots, l_{\pi_m}$. Contracting the expression

$$g_{j_1 i_1} g_{j_2 i_2} \dots g_{j_r i_r} g^{k_1 l_1} g^{k_2 l_2} \dots g^{k_s l_s} \delta^{i_{\lambda_1}}_{k_{\zeta_1}} \delta^{i_{\lambda_2}}_{k_{\zeta_2}} \dots \delta^{i_{\lambda_l}}_{k_{\zeta_l}} \delta^{j_{\theta_1}}_{l_{\pi_1}} \delta^{j_{\theta_2}}_{l_{\pi_2}} \dots \delta^{j_{\theta_m}}_{l_{\pi_m}}$$
$$= g_{j_1 i_1} g_{j_2 i_2} \dots g_{j_r i_r} \delta^{i_{\lambda_1}}_{k_{\zeta_1}} \delta^{i_{\lambda_2}}_{k_{\zeta_2}} \dots \delta^{i_{\lambda_l}}_{k_{\zeta_l}} g^{k_1 l_1} g^{k_2 l_2} \dots g^{k_s l_s} \delta^{j_{\theta_1}}_{l_{\pi_1}} \delta^{j_{\theta_2}}_{l_{\pi_2}} \dots \delta^{j_{\theta_m}}_{l_{\pi_m}}$$

in i_{λ_1}, i_{λ_2}, ..., i_{λ_l} yields

$$g_{j_{\lambda_1} i_{\lambda_1}} g^{k_{\zeta_1} l_{\zeta_1}} \delta^{i_{\lambda_1}}_{k_{\zeta_1}} g_{j_{\lambda_2} i_{\lambda_2}} g^{k_{\zeta_2} l_{\zeta_2}} \delta^{i_{\lambda_2}}_{k_{\zeta_2}} \dots g_{j_{\lambda_l} i_{\lambda_l}} g^{k_{\zeta_l} l_{\zeta_l}} \delta^{i_{\lambda_l}}_{k_{\zeta_l}} = \delta^{l_{\zeta_1}}_{j_{\lambda_1}} \delta^{l_{\zeta_2}}_{j_{\lambda_2}} \dots \delta^{l_{\zeta_l}}_{j_{\lambda_l}}.$$

The remaining expression in (6) should be contracted in l_{π_1}, l_{π_2}, ..., l_{π_m}. Contract the product

$$\delta^{j_{\theta_1}}_{l_{\rho_1}} \delta^{j_{\theta_2}}_{l_{\rho_2}} \dots \delta^{j_{\theta_m}}_{l_{\rho_m}} \delta^{l_{\zeta_1}}_{j_{\lambda_1}} \delta^{l_{\zeta_2}}_{j_{\lambda_2}} \dots \delta^{l_{\zeta_l}}_{j_{\lambda_l}}$$

in any possible way. Since $m < l$, the result of the contraction always contains at least one factor $\delta^{l_{\zeta_p}}_{j_{\lambda_p}}$. This factor, however, should be contracted with

$$V^{(\theta_1,\theta_2,\dots,\theta_m) j_{\Theta_{m+1}} j_{\Theta_{m+2}} \dots j_{\Theta_r}}_{(\pi_1,\pi_2,\dots,\pi_m) \qquad\qquad l_{\Pi_{m+1}} l_{\Pi_{m+2}} \dots l_{\Pi_r} l_{\Pi_{r+1}} \dots l_{\Pi_s}},$$

which results in a trace of

$$V^{(\theta_1,\theta_2,\dots,\theta_m)}_{(\pi_1,\pi_2,\dots,\pi_m)}.$$

But by hypothesis, this tensor is l-primitive, so its traces are all zero. This proves that $g(U,V) = 0$ as required.

Now if W is l-primitive and m-primitive, where $l \neq m$, we have $g(W,W) = 0$ hence $W = 0$. This completes the proof. □

The following is another version of the complete trace decomposition theorem.

Corollary 3. *Let $W \in T^r_s E$, and let $r \leq s$. There exist unique l-primitive tensors $W^{(l)} \in T^r_s E$, where $0 \leq l \leq r$, such that*

$$W = W^{(0)} + W^{(1)} + W^{(2)} + \dots + W^{(r)}. \tag{7}$$

Formula (7) is the *complete trace decomposition formula*. Every tensor belonging to $T^r_s E$ can be expressed in the form (7).

We are now concerned with the uniqueness of the $\delta^{(l)}$-components in (7).

Theorem 3. *Let $W \in T^r_s E$. If $r + s \leq n + 1$ then all $\delta^{(l)}$-components of W are unique.*

Proof. The simplest proof is an explicit one. Suppose that $r \leq s$. We prove that if $W^{(l)} = 0$, $1 \leq l \leq r$, then the $\delta^{(l)}$-component of W vanishes. According to (4), we must show that the homogeneous system

$$\sum_{\lambda_{(l,r)}} \sum_{\varphi_{(l,s)}} \sum_{\sigma \in \Sigma_l} \delta^{i_{\lambda_1}}_{k_{\varphi_{\sigma(1)}}} \delta^{i_{\lambda_2}}_{k_{\varphi_{\sigma(2)}}} \dots \delta^{i_{\lambda_l}}_{k_{\varphi_{\sigma(l)}}}$$

$$\cdot V^{(\lambda_1,\lambda_2,\dots,\lambda_l)}_{(\varphi_{\sigma(1)},\varphi_{\sigma(2)},\dots,\varphi_{\sigma(l)})}{}^{i_{\Lambda_{l+1}} i_{\Lambda_{l+2}} \dots i_{\Lambda_r}}{}_{k_{\vartheta_{l+1}} k_{\vartheta_{l+2}} \dots k_{\vartheta_r} k_{\vartheta_{r+1}} \dots k_{\vartheta_s}} = 0$$

has the trivial solution

$$V^{(\omega_1,\omega_2,\dots,\omega_l)}_{(\chi_{\mu(1)},\chi_{\mu(2)},\dots,\chi_{\mu(l)})} = 0 \tag{8}$$

only. Using any partitions $\varphi_{(l,r)} = \{\varphi_1, \varphi_2, \dots, \varphi_l\}$, $\Phi_{(l,r)} = \{\Phi_{l+1}, \Phi_{l+2}, \dots, \Phi_r\}$, $\psi_{(l,s)} = \{\psi_1, \psi_2, \dots, \psi_l\}$, $\Psi_{(l,s)} = \{\Psi_{l+1}, \Psi_{l+2}, ldots, \Psi_s\}$, and a permutation $\mu \in \Sigma_l$, we can prove that the tensor on the left in (8) is δ-generated. Since by hypothesis this tensor is traceless, this implies (8). □

4. The trace decomposition formula: Special cases

We give the trace decomposition formulas for tensor spaces of types $(1,1)$, $(1,2)$, $(1,3)$, $(2,2)$, and $(2,3)$ over E. Our first example is trivial, but is presented for the record: If $U \in T^1_1 E$, $U = U^i_k$, then there exist a unique traceless tensor $V = V^i_k$ and a unique number $c \in \mathbb{R}$ such that $U^i_k = V^i_k + c\delta^i_k$; c is given by $c(1/n)U^s_s$. Tensors of type $(1,4)$ were considered by Kovár by means of *Maple*; the resulting trace decomposition formula is very long (see [9], [14]).

Let us discuss tensors of type $(1,2)$ (the *torsion tensors* in differential geometry of connections).

Theorem 4. *Let $U \in T^1_2 E$, $U = U^i_{kl}$. There exist a unique traceless tensor $V \in T^1_2 E$, $V = V^i_{kl}$, and unique tensors $P, Q \in T^0_1 E$, $P = P_k$, $Q = Q_k$, such that*

$$U^i_{kl} = V^i_{kl} + \delta^i_k P_l + \delta^i_l Q_k.$$

These tensors are given by

$$P_l = \frac{1}{n^2-1}(nU^t_{tl} - U^t_{lt}), \quad Q_k = \frac{1}{n^2-1}(-U^t_{tk} + nU^t_{kt}).$$

Note that the tensor V is already *defined* by the tensors P, Q, and U.

Now consider tensors of type $(1,3)$ and $(2,2)$ (*curvature tensors* describing properties of connections and metric fields on smooth manifolds). Of particular importance are *traceless* tensors of this type, the Weyl tensors; as a consequence of the trace decomposition theory, they can be defined as traceless components of tensors of type $(1,3)$, satisfying additional symmetry properties. For more discussion, we refer to Krupka [12], [13]. Note that the presented formulas illustrate possible non-uniqueness of the trace decomposition.

Theorem 5. *Let $U \in T^1_3 E$, $U = U^i_{klm}$.*

(a) *Suppose that $n \geq 3$. Then there exist a unique traceless tensor $V \in T^1_3 E$, $V = V^i_{klm}$, and unique tensors $P, Q, R \in T^0_2 E$, $P = P_{lm}$, $Q = Q_{km}$, $R = R_{kl}$, such that*

$$U^i_{klm} = V^i_{klm} + \delta^i_k P_{lm} + \delta^i_l Q_{km} + \delta^i_m R_{kl}.$$

These tensors are given by

$$P_{kl} = \frac{1}{(n^2-1)(n^2-4)}\Big(n(n^2-3)U^t_{tkl} - (n^2-2)U^t_{ktl}$$
$$+ nU^t_{klt} - 2U^t_{tlk} + nU^t_{ltk} - (n^2-2)U^t_{lkt}\Big),$$
$$Q_{kl} = \frac{1}{(n^2-1)(n^2-4)}\Big(-(n^2-2)U^t_{tkl} + n(n^2-3)U^t_{ktl}$$
$$- (n^2-2)U^t_{klt} + nU^t_{tlk} - 2U^t_{ltk} + nU^t_{lkt}\Big),$$
$$R_{kl} = \frac{1}{(n^2-1)(n^2-4)}\Big(nU^t_{tkl} - (n^2-2)U^t_{ktl} + n(n^2-3)U^t_{klt}$$
$$- (n^2-2)U^t_{tlk} + nU^t_{ltk} - 2U^t_{lkt}\Big).$$

(b) *Suppose that* $n = 2$. *Then there exist a unique traceless tensor* $V \in T^1_3E$, $V = V^i_{klm}$, *and tensors* $P, Q, R \in T^0_2E$, $P = P_{kl}$, $Q = Q_{kl}$, $R = R_{kl}$, *such that*

$$U^i_{klm} = V^i_{klm} + \delta^i_k P_{lm} + \delta^i_l Q_{km} + \delta^i_m R_{kl}.$$

These tensors are given by

$$P_{11} = \frac{1}{4}\,(3U^t_{t11} - U^t_{1t1} - U^t_{11t}),$$
$$P_{12} = \frac{5}{6}\,(U^t_{t12} - \mu) - \frac{2}{3}\,U^t_{1t2} + \frac{1}{2}\,U^t_{12t} - \frac{1}{3}\,U^t_{t21} + \frac{5}{6}\,(U^t_{2t1} - \mu),$$
$$P_{21} = -\frac{1}{3}\,(U^t_{t12} - \mu) + \frac{2}{3}\,U^t_{1t2} - U^t_{12t} + \frac{4}{3}\,U^t_{t21} - \frac{2}{3}\,(U^t_{2t1} - \mu),$$
$$P_{22} = \frac{1}{4}\,(3U^t_{t22} - U^t_{2t2} - U^t_{22t}),$$
$$Q_{11} = \frac{1}{4}\,(-U^t_{t11} + 3U^t_{1t1} - U^t_{11t}),$$
$$Q_{12} = -\frac{2}{3}\,(U^t_{t12} - \mu) + \frac{4}{3}\,U^t_{1t2} - U^t_{12t} + \frac{2}{3}\,U^t_{t21} - \frac{1}{3}\,(U^t_{2t1} - \mu),$$
$$Q_{21} = \frac{1}{6}\,(U^t_{t12} - \mu) - \frac{1}{3}\,U^t_{1t2} + \frac{1}{2}\,U^t_{12t} - \frac{2}{3}\,U^t_{t21} + \frac{5}{6}\,(U^t_{2t1} - \mu),$$
$$Q_{22} = \frac{1}{4}\,(-U^t_{t22} + 3U^t_{2t2} - U^t_{22t}),$$
$$R_{11} = \frac{1}{4}\,(-U^t_{t11} - U^t_{1t1} + 3U^t_{11t}),$$
$$R_{12} = \frac{1}{2}\,(U^t_{t12} - \mu) - U^t_{1t2} + \frac{3}{2}\,U^t_{12t} - U^t_{t21} + \frac{1}{2}\,(U^t_{2t1} - \mu),$$
$$R_{21} = \mu,$$
$$R_{22} = \frac{1}{4}\,(-U^t_{t22} - U^t_{2t2} + 3U^t_{22t}),$$

where $\mu \in \mathbb{R}$ *is a parameter.*

Theorem 6. *Let* $U \in T^2_2 E$, $U = U^{ij}_{kl}$.

(a) *Suppose that* $n \geq 3$. *Then there exist a unique traceless tensor* $V \in T^2_2 E$, $V = V^{ij}_{kl}$, *unique traceless tensors* $P, Q, R, S \in T^1_1 E$, $P = P^i_k$, $Q = Q^i_k$, $R = R^i_k$, $S = S^i_k$, *and unique numbers* $G, H \in \mathbb{R}$, *such that*

$$U^{ij}_{kl} = V^{ij}_{kl} + \delta^i_k P^j_l + \delta^i_l Q^j_k + \delta^j_k R^i_l + \delta^j_l S^i_k + \delta^i_k \delta^j_l G + \delta^i_l \delta^j_k H.$$

These tensors are given by

$$P^i_j = \frac{1}{n(n^2-4)}((n^2-2)U^{si}_{sj} - nU^{si}_{js} - nU^{is}_{sj} + 2U^{is}_{js} - n\delta^i_j U^{st}_{st} + 2\delta^i_j U^{st}_{ts}),$$

$$Q^i_j = \frac{1}{n(n^2-4)}(-nU^{si}_{sj} + (n^2-2)U^{si}_{js} + 2U^{is}_{sj} - nU^{is}_{js} + 2\delta^i_j U^{st}_{st} - n\delta^i_j U^{st}_{ts}),$$

$$R^i_j = \frac{1}{n(n^2-4)}(-nU^{si}_{sj} + 2U^{si}_{js} + (n^2-2)U^{is}_{sj} - nU^{is}_{js} + 2\delta^i_j U^{st}_{st} - n\delta^i_j U^{st}_{ts}),$$

$$S^i_j = \frac{1}{n(n^2-4)}(2U^{si}_{sj} - nU^{si}_{js} - nU^{is}_{sj} + (n^2-2)U^{is}_{js} + 2\delta^i_j U^{st}_{ts} - n\delta^i_j U^{st}_{st}),$$

$$G = \frac{1}{n(n^2-1)}(nU^{st}_{st} - U^{st}_{ts}),$$

$$H = \frac{1}{n(n^2-1)}(-U^{st}_{st} + nU^{st}_{ts}).$$

(b) *Suppose that* $n = 2$. *Then there exist a unique traceless tensor* $V \in T^2_2 E$, $V = V^{ij}_{kl}$, *traceless tensors* $P, Q, R, S \in T^1_1 E$, $P = P^i_k$, $Q = Q^i_k$, $R = R^i_k$, $S = S^i_k$, *and unique numbers* $G, H \in R$ *such that*

$$U^{ij}_{kl} = V^{ij}_{kl} + \delta^i_k P^j_l + \delta^i_l Q^j_k + \delta^j_k R^i_l + \delta^j_l S^i_k + \delta^i_k \delta^j_l G + \delta^i_l \delta^j_k H.$$

These tensors are given by

$$P^i_j = U^{si}_{sj} - \frac{1}{2}\,U^{si}_{js} - \frac{1}{2}\,U^{is}_{sj} - \frac{1}{2}\,\delta^i_j U^{st}_{st} + \frac{1}{2}\,\delta^i_j U^{st}_{ts} + \mu^i_j,$$

$$Q^i_j = -\frac{1}{2}\,U^{si}_{sj} + \frac{3}{4}\,U^{si}_{js} + \frac{1}{4}\,U^{is}_{sj} + \frac{1}{4}\,\delta^i_j U^{st}_{st} - \frac{1}{2}\,\delta^i_j U^{st}_{ts} - \mu^i_j,$$

$$R^i_j = -\frac{1}{2}\,U^{si}_{sj} + \frac{1}{4}\,U^{si}_{js} + \frac{3}{4}\,U^{is}_{sj} + \frac{1}{4}\,\delta^i_j U^{st}_{st} - \frac{1}{2}\,\delta^i_j U^{st}_{ts} - \mu^i_j,$$

$$S^i_j = \mu^i_j,$$

$$G = \frac{1}{6}\,(2U^{st}_{st} - U^{st}_{ts}),$$

$$H = \frac{1}{6}\,(-U^{st}_{st} + 2U^{st}_{ts}),$$

where μ^i_j *are real parameters such that* $\mu^1_1 + \mu^2_2 = 0$.

In Section 3, Theorem 3, we have shown that the condition $r + s \leq n + 1$ is sufficient for the uniqueness of the complete trace decomposition of the tensor space $T^r_s E$. The following result shows, in particular, that for $n = 3$, this condition is not necessary. The proof consists in finding an explicit form of the trace decomposition equations, and determining their rank by a direct computation of the corresponding determinants.

Theorem 7. *Let $U \in T^2_3 E$, $U = U^{k_1k_2}{}_{j_1j_2j_3}$, and suppose that $n \geq 3$. Then there exist a unique traceless tensor $V \in T^2_3 E$, $V = V^{k_1k_2}{}_{j_1j_2j_3}$, unique traceless tensors $V^{(\alpha)}_{(\beta)} \in T^1_2 E$, $V^{(\alpha)}_{(\beta)} = V^{(\alpha)i}_{(\beta)\,jk}$, where $\alpha = 1,2$, $\beta = 1,2,3$, and unique tensors $V^{(1,2)}_{(1,2)}$, $V^{(1,2)}_{(2,1)}$, $V^{(1,2)}_{(1,3)}$, $V^{(1,2)}_{(3,1)}$, $V^{(1,2)}_{(2,3)}$, $V^{(1,2)}_{(3,2)} \in T^0_1 E$, $V^{(\mu,\nu)}_{(\sigma,\tau)} = V^{(\mu,\nu)}_{(\sigma,\tau)\,k}$ such that*

$$\begin{aligned}
U^{k_1k_2}{}_{j_1j_2j_3} &= V^{k_1k_2}{}_{j_1j_2j_3} + \delta^{k_1}_{j_1} V^{(1)k_2}_{(1)}{}_{j_2j_3} + \delta^{k_1}_{j_2} V^{(1)k_2}_{(2)}{}_{j_1j_3} + \delta^{k_1}_{j_3} V^{(1)k_2}_{(3)}{}_{j_1j_2} \\
&+ \delta^{k_2}_{j_1} V^{(2)k_1}_{(1)}{}_{j_2j_3} + \delta^{k_2}_{j_2} V^{(2)k_1}_{(2)}{}_{j_1j_3} + \delta^{k_2}_{j_3} V^{(2)k_1}_{(3)}{}_{j_1j_2} \\
&+ \delta^{k_1}_{j_1}\delta^{k_2}_{j_2} V^{(1,2)}_{(1,2)\,j_3} + \delta^{k_1}_{j_2}\delta^{k_2}_{j_1} V^{(1,2)}_{(2,1)\,j_3} + \delta^{k_1}_{j_1}\delta^{k_2}_{j_3} V^{(1,2)}_{(1,3)\,j_2} \\
&+ \delta^{k_1}_{j_3}\delta^{k_2}_{j_1} V^{(1,2)}_{(3,1)\,j_2} + \delta^{k_1}_{j_2}\delta^{k_2}_{j_3} V^{(1,2)}_{(2,3)\,j_1} + \delta^{k_1}_{j_3}\delta^{k_2}_{j_2} V^{(1,2)}_{(3,2)\,j_1}.
\end{aligned}$$

5. Symmetric-antisymmetric tensors

Let E be an n-dimensional vector space. A tensor $U \in T^r_s E$ is said to be *symmetric-antisymmetric*, if it is symmetric in all superscripts and antisymmetric in all subscripts. We denote the subspace of symmetric-antisymmetric tensors in $T^r_s E$ by $\mathrm{Z}^{(r)}_{(s)} E$; we wish to present in this section the *trace decomposition formula* for these tensors.

Let $U \in \mathrm{Z}^{(r)}_{(s)} E$ be a tensor. We set

$$C_{r,s} = \frac{(r+1)(s+1)}{n+r-s},$$

and

(1) $$\mathrm{q}\, U = C_{r,s} U^{i_2i_3\dots i_{r+1}}{}_{j_2j_3\dots j_{s+1}} \delta^{i_1}_{j_1} \quad \mathrm{alt}\,(j_1j_2\dots j_{s+1}) \quad \mathrm{sym}\,(i_1i_2\dots i_{r+1}),$$

where alt (respectively sym) means *alternation* (respectively *symmetrization*) in the indicated indices, and

(2) $$\mathrm{tr}\, U = \mathrm{tr}\,{}^{(1)}_{(1)} U = U^{pi_1i_2\dots i_{r-1}}{}_{pj_1j_2\dots j_{s-1}}.$$

The following two theorems summarize basic properties of the linear mappings $\mathrm{q} : \mathrm{Z}^{(r)}_{(s)} E \longrightarrow \mathrm{Z}^{(r+1)}_{(s+1)} E$ and $\mathrm{tr} : \mathrm{Z}^{(r)}_{(s)} E \longrightarrow \mathrm{Z}^{(r-1)}_{(s-1)} E$, and of linear equations associated with them.

Theorem 8. *Every tensor $U \in \mathrm{Z}^{(r)}_{(s)} E$ satisfies $\mathrm{q}\,\mathrm{q}\, U = 0$, $\mathrm{tr}\,\mathrm{tr}\, U = 0$, and*

(3) $$U = \mathrm{q}\,\mathrm{tr}\, U + \mathrm{tr}\,\mathrm{q}\, U.$$

Proof. Theorem 1 can be proved by explicit computations. □

Theorem 9. *Let $U \in \mathrm{Z}^{(r)}_{(s)} E$.*

(a) *Equation $\mathrm{q}\, V + \mathrm{tr}\, W = U$ for unknown tensors $V \in \mathrm{Z}^{(r-1)}_{(s-1)} E$, $W \in \mathrm{Z}^{(r+1)}_{(s+1)} E$ has a unique solution such that $\mathrm{tr}\, V = 0$, $\mathrm{q}\, W = 0$. This solution is given by $V = \mathrm{tr}\, U$, $W = \mathrm{q}\, U$.*

(b) *Equation $\mathrm{q}\, X = U$ has a solution $X \in \mathrm{Z}^{(r-1)}_{(s-1)} E$ if and only if $\mathrm{q}\, U = 0$. If this* *condition is satisfied, then* $X = \operatorname{tr} U$ *is a solution. Any other solution is of the form* $X' = X + \operatorname{q} Y$ *for some tensor* $Y \in \mathrm{Z}^{(r-12)}_{(s-2)} E$.

Remark 4. Formula (3) is the *complete trace decomposition formula* of U. This formula has a remarkable structure: It shows that formally, the operators q and tr have the same properties as the *homotopy operator* in the integrability theory of differential forms on smooth manifolds.

Remark 5. Equations of the form $\operatorname{tr} X = U$ can be solved in the same way as the equation $\operatorname{q} X = U$.

6. Elementary symmetrization operators

In this section, we use multi-indices of the form $J = (j_1 j_2 \dots j_r)$, where $r \geq 1$, $1 \leq j_1, j_2, \dots, j_r \leq n$; the length of J is defined to be $|J| = r$. We consider contravariant tensors $U = U^{J_1 J_2 \dots J_p}$, *symmetric* in the superscripts entering each multi-index; the *characteristic* of U, $\operatorname{char} U$, is defined to be the p-tuple $(r_1, r_2, \dots, r_p)$, where $|J_i| = r_i$. The lengths of different multi-indices do not necessarily coincide. The vector subspace of tensors of characteristic $(r_1, r_2, \dots, r_p)$ in $T^{r+r_2+\dots+r_p} E$ is denoted $\mathrm{Z}^{(r_1, r_2, \dots, r_p)} E$. The tensors belonging to the tensor space $\mathrm{Z}^{(r_1, r_2, \dots, r_p)} E$ are called p-*multisymmetric*; if $p = 1$, we speak of symmetric tensors.

We introduce linear mappings between multisymmetric tensor spaces, representing *successive symmetrizations* in different tensor indices, and study their spectral properties. These mappings appear in the trace decomposition formulas, and will be used in next sections. It is sufficient to define them for 2-multisymmetric tensors. In this case we usually use the standard index notation which is more explicit; we write $U = U^{j_1 j_2 \dots j_l \, k_1 k_2 \dots k_m}$ for 2-multisymmetric tensors of characteristic (l, m), where $l, m \geq 0$. All formulas can be easily written for general multisymmetric tensors (i.e., for any two multi-indices labelling these tensors).

Let $U \in \mathrm{Z}^{(l,m)} E$, $U = U^{j_1 j_2 \dots j_l \, k_1 k_2 \dots k_m}$, be a tensor. We assign to U new tensors $\operatorname{sym}_{1,2} U \in \mathrm{Z}^{(l+1,m-1)} E$ and $\operatorname{sym}_{2,1} U \in \mathrm{Z}^{(l-1,m+1)} E$ by

$$
\begin{aligned}
&\operatorname{sym}_{1,2} U = U^{j_2 j_3 \dots j_l j_{l+1} \, j_1 k_1 k_2 \dots k_{m-1}} \quad \operatorname{sym}(j_1 j_2 \dots j_l j_{l+1}),\\
&\operatorname{sym}_{2,1} U = U^{k_1 j_1 j_2 \dots j_{l-1} \, k_2 k_3 \dots k_m k_{m+1}} \quad \operatorname{sym}(k_1 k_2 \dots k_m k_{m+1}).
\end{aligned} \tag{1}
$$

In the symmetrizations the usual symmetrization coefficients are included which are equal in the considered case to $1/(l+1)$ and $1/(m+1)$, respectively.

Note that $\operatorname{char} \operatorname{sym}_{1,2} U = (l+1, m-1)$, $\operatorname{char} \operatorname{sym}_{2,1} U = (l-1, m+1)$, and

$$
\operatorname{char} \operatorname{sym}^p_{1,2} U = (l+p, m-p), \quad \operatorname{char} \operatorname{sym}^p_{2,1} U = (l-p, m+p) \tag{2}
$$

whenever the operators $\operatorname{sym}^p_{1,2}$, $\operatorname{sym}^p_{2,1}$ are defined. Composing the operator $\operatorname{sym}^p_{1,2}$ several times, we obtain

$$
\begin{aligned}
\operatorname{sym}^p_{1,2} U = U^{j_{m+1} j_{m+2} \dots j_{m+l} \, j_1 j_2 \dots j_m}{}_{i_1 i_2 \dots i_s} \quad &\operatorname{sym}(j_1 j_2 j_3 \dots j_l j_{l+1})\\
\operatorname{sym}(j_1 j_2 j_3 \dots j_l j_{l+1}) \quad \dots \quad &\operatorname{sym}(j_1 j_2 j_3 \dots j_l j_{l+1} j_{l+2} \dots j_{l+p}).
\end{aligned}
$$

Thus, the powers of $\operatorname{sym}^p_{1,2}$ satisfy

$$
\operatorname{sym}^m_{1,2} U = U^{j_{m+1} j_{m+2} \dots j_{m+l} \, j_1 j_2 \dots j_m}{}_{i_1 i_2 \dots i_s} \quad \operatorname{sym}(j_1 j_2 j_3 \dots j_l j_{l+1} \dots j_{l+m}).
$$

We note that $\operatorname{sym}^p_{1,2}U$ (respectively $\operatorname{sym}^p_{2,1}U$) is defined only for $p \le m$ (respectively $p \le l$). $\operatorname{sym}^m_{1,2}U$ (respectively $\operatorname{sym}^l_{2,1}U$) is a *symmetric* tensor of characteristic $(l+m, 0, s)$ (respectively $(0, m+l, s)$). The operators $\operatorname{sym}^l_{1,2}\operatorname{sym}^l_{2,1}$ and $\operatorname{sym}^m_{2,1}\operatorname{sym}^m_{1,2}$ are equal to *the complete symmetrization* in the space of tensors of characteristic (l, m), $\mathrm{Z}^{(l,m)}E$. In particular, $\operatorname{sym}^l_{1,2}\operatorname{sym}^l_{2,1} = \operatorname{sym}^m_{2,1}\operatorname{sym}^m_{1,2}$, and, since the complete symmetrization is a projector, $\operatorname{sym}^l_{1,2}\operatorname{sym}^l_{2,1}\operatorname{sym}^l_{1,2}\operatorname{sym}^l_{2,1} = \operatorname{sym}^l_{1,2}\operatorname{sym}^l_{2,1}$, and $\operatorname{sym}^m_{2,1}\operatorname{sym}^m_{1,2}\operatorname{sym}^m_{2,1}\operatorname{sym}^m_{1,2} = \operatorname{sym}^m_{2,1}\operatorname{sym}^m_{1,2}$.

We also introduce slightly modified symmetrization operators, differing from (1) by a numerical factor,

$$\sigma = (l+1)\operatorname{sym}_{1,2}, \qquad \tau = (m+1)\operatorname{sym}_{2,1}. \tag{3}$$

We call these operators the *elementary symmetrization operators*. Note that

$$\tau^k U = \frac{(m+k)!}{m!}\operatorname{sym}^k_{2,1}U,$$

and

$$\sigma^k\tau^k U = \frac{(m+k)!}{m!}\frac{l!}{(l-k)!}\operatorname{sym}^k_{1,2}\operatorname{sym}^k_{2,1}U.$$

For $k = l$, the complete symmetrization operator $\operatorname{sym}^l_{1,2}\operatorname{sym}^l_{2,1}$ satisfies

$$\sigma^l\tau^l U = \tau^m\sigma^m U = \frac{l!\,(m+l)!}{m!}\operatorname{sym}^l_{1,2}\operatorname{sym}^l_{2,1}U = \frac{m!\,(m+l)!}{l!}\operatorname{sym}^m_{2,1}\operatorname{sym}^m_{1,2}U. \tag{4}$$

Lemma 1. *The operators σ and τ satisfy the commutation relation*

$$\tau\sigma U - \sigma\tau U = (m-l)U. \tag{5}$$

Proof. This is an immediate consequence of an explicit formula

$$\begin{aligned}
\operatorname{sym}_{2,1}\operatorname{sym}_{1,2}U &= \frac{1}{l+1}U^{j_1j_2\dots j_l\ k_1k_2\dots k_m} \\
&+ \frac{1}{m(l+1)}(U^{k_1j_lj_2\dots j_{l-1}\ j_1k_mk_2k_3\dots k_{m-1}} + U^{k_1j_1j_lj_3\dots j_{l-2}j_{l-1}\ j_2k_mk_2k_3\dots k_{m-1}} \\
&+ \dots + U^{k_1j_1j_2\dots j_{l-2}j_l\ j_{l-1}k_mk_2k_3\dots k_{m-1}} + U^{k_1j_1j_2\dots j_{l-2}j_{l-1}\ j_lk_mk_2k_3\dots k_{m-1}} \\
&+ U^{k_2j_lj_2\dots j_{l-1}\ j_1k_mk_1k_3\dots k_{m-1}} + U^{k_2j_1j_lj_3\dots j_{l-2}j_{l-1}\ j_2k_mk_1k_3\dots k_{m-1}} \\
&+ \dots + U^{k_2j_1j_2\dots j_{l-2}j_l\ j_{l-1}k_mk_1k_3\dots k_{m-1}} + U^{k_2j_1j_2\dots j_{l-2}j_{l-1}\ j_lk_mk_1k_3\dots k_{m-1}} \\
&+ \dots + U^{k_mj_lj_2\dots j_{l-1}\ j_1k_1k_2k_3\dots k_{m-1}} + U^{k_mj_1j_lj_3\dots j_{l-2}j_{l-1}\ j_2k_1k_2k_3\dots k_{m-1}} \\
&+ \dots + U^{k_mj_1j_2\dots j_{l-2}j_l\ j_{l-1}k_1k_2k_3\dots k_{m-1}} + U^{k_mj_1j_2\dots j_{l-2}j_{l-1}\ j_lk_1k_2k_3\dots k_{m-1}}),
\end{aligned} \tag{6}$$

which implies

$$\begin{aligned}
&m(l+1)\operatorname{sym}_{2,1}\operatorname{sym}_{1,2}U - mU^{j_1j_2\dots j_l\ k_1k_2\dots k_m} \\
&\quad = l(m+1)\operatorname{sym}_{1,2}\operatorname{sym}_{2,1}U - lU^{j_1j_2\dots j_l\ k_1k_2\dots k_m}. \square
\end{aligned}$$

We find relations between the powers $(\sigma\tau)^p$ and the composite operators $\sigma^p\tau^p$. For tensors of characteristic (l, m), put

$$N_k = (k+1)(m-l+k)$$

for all $k = 0, 1, 2, \ldots, l-1$. Note that $N_{k+1} - N_k = m - l + 2(k+1)$, and

$$2 + m - l \leq m - l + 2(k+1) \leq m - l + 2l = m + l.$$

If $m \geq l$, then N_k will be an increasing sequence of positive numbers. Denote by Σ_q^p, where $0 \leq p \leq l$, $0 \leq q \leq p$, the double sequence of *symmetric polynomials* in the variables N_k. These polynomials are defined by

$$\Sigma_1^0 = 0, \quad \Sigma_0^p = 1, \tag{7}$$

and for every p, $1 \leq p \leq l$,

$$\begin{aligned} &\Sigma_1^p = \Sigma_1^{p.-1} + \Sigma_0^{p-1} N_{p-1}, \\ &\Sigma_2^p = \Sigma_2^{p-1} + \Sigma_1^{p-1} N_{p-1}, \\ &\cdots \\ &\Sigma_{p-1}^p = \Sigma_{p-1}^{p-1} + \Sigma_{p-2}^{p-1} N_{p-1}, \\ &\Sigma_p^p = \Sigma_{p-1}^{p-1} N_{p-1}. \end{aligned} \tag{8}$$

Note that both N_k and Σ_q^p depend on the characteristic (l, m) of the underlying tensor space; in fact, Σ_q^p is a function of the difference $m - l$. To express this dependence, we sometimes write

$$N_k = N_k^{(m-l)}, \quad \Sigma_q^p = \Sigma_q^{(m-l),p}. \tag{9}$$

Theorem 10. *For any tensor U of characteristic (l, m), and all $k = 0, 1, 2, \ldots, l-1$,*

$$\begin{aligned} \sigma^k \tau^k U = &(-1)^0 \Sigma_0^k (\tau\sigma)^k U + (-1)^1 \Sigma_1^k (\tau\sigma)^{k-1} U + (-1)^2 \Sigma_2^k (\tau\sigma)^{k-2} U \\ &+ \ldots + (-1)^{k-1} \Sigma_{k-1}^k (\tau\sigma) U + (-1)^k \Sigma_k^k U. \end{aligned} \tag{10}$$

Proof. From Lemma 1, (5) it follows that $\tau^2\sigma U - \tau\sigma\tau U = (m-l)\tau U$ and, since $\operatorname{char} \tau U = (l-1, m+1)$ and $\tau\sigma\tau U = \sigma\tau^2 U + (m-l+2)\tau U$, we have

$$\tau^2\sigma U - \sigma\tau^2 U = (m-l)\tau U + (m-l+2)\tau U = 2(m-l+1)\tau U. \tag{11}$$

We apply τ to (11); we get $\tau^3\sigma U - \tau\sigma\tau^2 U = 2(m-l+1)\tau^2 U$. But the characteristic $\operatorname{char} \tau^2 U$ is $\operatorname{char} \tau^2 U = (l-2, m+2)$, so we have $\tau\sigma\tau^2 U = \sigma\tau^3 U + (m-l+4)\tau^2 U$, and $\tau^3\sigma U - \sigma\tau^3 U = 3(m-l+2)\tau^2 U$. Now it is easy to prove by induction that for all $k = 0, 1, 2, \ldots, l$

$$\tau^k \sigma U - \sigma\tau^k U = N_{k-1} \tau^{k-1} U. \tag{12}$$

Apply σ^k to (12). We have $\sigma^k \tau^{k+1} \sigma U - \sigma^{k+1} \tau^{k+1} U = N_k \sigma^k \tau^k U$, i.e.,

$$\sigma^{k+1} \tau^{k+1} U = \sigma^k \tau^k (\tau\sigma U - N_k U). \tag{13}$$

Using formula (13) repeatedly, we can express $\sigma^{k+1}\tau^{k+1}$ as the sum of powers of $\tau\sigma$. For $k = 0, 1, 2$ we obtain

$$\begin{aligned} &\sigma\tau U = \tau\sigma U - \Sigma_1^1 U, \\ &\sigma^2\tau^2 U = (\tau\sigma)^2 U - \Sigma_1^2 \tau\sigma U + \Sigma_2^2 U, \\ &\sigma^3\tau^3 U = (\tau\sigma)^3 U - \Sigma_1^3 (\tau\sigma)^2 U + \Sigma_2^3 \tau\sigma U - \Sigma_3^3 U. \end{aligned}$$

On induction one can prove that

$$\sigma^k\tau^k U = (-1)^0\Sigma_0^k(\tau\sigma)^k U + (-1)^1\Sigma_1^k(\tau\sigma)^{k-1}U + (-1)^2\Sigma_2^k(\tau\sigma)^{k-2}U + \ldots + (-1)^{k-1}\Sigma_{k-1}^k\tau\sigma U + (-1)^k\Sigma_k^k U.$$

for all k. □

In formula (10), we can express explicitly the dependence of the coefficients on the characteristic of U (see (9)). We get

$$\begin{aligned}\sigma^k\tau^k U &= (-1)^0\Sigma_0^{(m-l),k}(\tau\sigma)^k U + (-1)^1\Sigma_1^{(m-l),k}(\tau\sigma)^{k-1}U \\ &+ (-1)^2\Sigma_2^{(m-l),k}(\tau\sigma)^{k-2}U + \ldots + (-1)^{k-1}\Sigma_{k-1}^{(m-l),k}(\tau\sigma)U \\ &+ (-1)^k\Sigma_k^{(m-l),k}U.\end{aligned} \tag{14}$$

This dependence is essential in the following formula for the decomposition of the operator $\sigma^k\tau^k$.

Corollary 4. *For any tensor U of characteristic (l,m), and all $k = 0,1,2,\ldots,l$,*

$$\begin{aligned}\sigma^k\tau^k U &= (-1)^0\Sigma_0^{(m-l+2),k-1}(\sigma\tau)^k U \\ &+ (-1)^1\Sigma_1^{(m-l+2),k-1}(\sigma\tau)^{k-1}U + (-1)^2\Sigma_2^{(m-l+2),k-1}(\sigma\tau)^{k-2}U \\ &+ \ldots + (-1)^{k-2}\Sigma_{k-2}^{(m-l+2),k-1}(\sigma\tau)^2 U + (-1)^{k-1}\Sigma_{k-1}^{(m-l+2),k-1}\sigma\tau U.\end{aligned} \tag{15}$$

Proof. To prove (15), we apply (10) to tensors of the form τU. Since the characteristic of these tensors is $(l-1, m+1)$, we have to use the polynomials $\Sigma_q^{(m-l+2),p}$ instead of $\Sigma_q^{(m-l),p}$. Thus,

$$\begin{aligned}\sigma^{k-1}\tau^{k-1}\tau U &= (-1)^0\Sigma_0^{(m-l+2),k-1}(\tau\sigma)^{k-1}\tau U \\ &+ (-1)^1\Sigma_1^{(m-l+2),k-1}(\tau\sigma)^{k-2}\tau U + (-1)^2\Sigma_2^{(m-l+2),k-1}(\tau\sigma)^{k-3}\tau U \\ &+ \ldots + (-1)^{k-2}\Sigma_{k-2}^{(m-l+2),k-1}(\tau\sigma)\tau U + (-1)^{k-1}\Sigma_{k-1}^{(m-l+2),k-1}\tau U.\end{aligned}$$

Acting on (14) with σ, we get

$$\begin{aligned}\sigma^k\tau^k U &= (-1)^0\Sigma_0^{(m-l+2),k-1}\sigma(\tau\sigma)^{k-1}\tau U \\ &+ (-1)^1\Sigma_1^{(m-l+2),k-1}\sigma(\tau\sigma)^{k-2}\tau U + (-1)^2\Sigma_2^{(m-l+2),k-1}\sigma(\tau\sigma)^{k-3}\tau U \\ &+ \ldots + (-1)^{k-2}\Sigma_{k-2}^{(m-l+2),k-1}\sigma(\tau\sigma)\tau U + (-1)^{k-1}\Sigma_{k-1}^{(m-l+2),k-1}\sigma\tau U.\end{aligned}$$

Now (15) follows from the equality $\sigma(\tau\sigma)^{k-1}\tau U = (\sigma\tau)^k U$. □

Lemma 2. (a) *The operators $(\tau\sigma)^p$ and $\sigma^k\tau^k$ satisfy*

$$(\tau\sigma)^p\sigma^k\tau^k U = \sigma^k\tau^k(\tau\sigma)^p U, \quad (\sigma\tau)^p\sigma^k\tau^k U = \sigma^k\tau^k(\sigma\tau)^p U. \tag{16}$$

(b) *$\sigma^l\tau^l$ satisfies*

$$\sigma^l\tau^l\sigma^l\tau^l U = \frac{l!\,(m+l)!}{m!}\sigma^l\tau^l U, \tag{17}$$

and

$$(\tau\sigma)^k\sigma^l\tau^l U = \sigma^l\tau^l(\tau\sigma)^k U = m^k(l+1)^k\sigma^l\tau^l U. \tag{18}$$

Proof. Formulas (16) are immediate consequences of (10) and (15), and (17) follows from (4). To derive (18) note that by Lemma2, (a), $(\tau\sigma)\sigma^l\tau^l U = \sigma^l\tau^l(\tau\sigma)U$. But from (6),

$$\begin{aligned}&\mathrm{sym}\,^l_{1,2}\mathrm{sym}\,^l_{2,1}\mathrm{sym}\,_{2,1}\mathrm{sym}\,_{1,2}U\\ &= \left(\frac{1}{l+1}+\frac{lm}{m(l+1)}\right)\mathrm{sym}\,^l_{1,2}\mathrm{sym}\,^l_{2,1}U = \mathrm{sym}\,^l_{1,2}\mathrm{sym}\,^l_{2,1}U.\end{aligned}$$

Thus, using (3) and (4), $(\tau\sigma)\sigma^l\tau^l U = \sigma^l\tau^l(\tau\sigma)U = m(l+1)\sigma^l\tau^l U$. Repeating this formula k times we get (18). □

Now consider the operator $\sigma^k\tau^k : \mathrm{Z}^{(l,m)}E \longrightarrow \mathrm{Z}^{(l,m)}E$ for $k = l$; note that this operator satisfies (4). We begin by proving an identity involving the symmetric polynomials Σ^p_q (7), (8).

Theorem 11. *The polynomials* $\Sigma^l_1, \Sigma^l_2, \dots, \Sigma^l_l$ *satisfy*

$$\begin{aligned}&(-1)^0\Sigma^l_0 m^l(l+1)^l + (-1)^1\Sigma^l_1 m^{l-1}(l+1)^{l-1} + (-1)^2\Sigma^l_2 m^{l-2}(l+1)^{l-2}\\ &\quad + \dots + (-1)^{l-1}\Sigma^l_{l-1}m(l+1) + (-1)^l\Sigma^l_l = \frac{l!\,(m+l)!}{m!}.\end{aligned} \tag{19}$$

Proof. To derive (19), we write formula (10) for $k = l$. We have

$$\begin{aligned}\sigma^l\tau^l U &= (-1)^0\Sigma^l_0(\tau\sigma)^l U + (-1)^1\Sigma^l_1(\tau\sigma)^{l-1}U + (-1)^2\Sigma^l_2(\tau\sigma)^{l-2}U\\ &\quad + \dots + (-1)^{l-1}\Sigma^l_{l-1}(\tau\sigma)U + (-1)^l\Sigma^l_l U.\end{aligned}$$

(19) now follows from (17), (18). □

Let U be a given non-zero tensor of characteristic (l, m), and let $\lambda \in \mathbb{R}$ be a non-zero real number. We now study the equation

$$X - \frac{1}{\lambda}\tau\sigma X = U \tag{20}$$

for an unknown tensor $X \in \mathrm{Z}^{(l,m)}E$. We denote

$$\mathcal{S}^{(l,m)}_\lambda X = X - \frac{1}{\lambda}\tau\sigma X. \tag{21}$$

$\mathcal{S}^{(l,m)}_\lambda$ is a linear operator on the tensor space $\mathrm{Z}^{(l,m)}E$. We want to find all λ such that there exists the *inverse operator* $\mathcal{T}^{(l,m)}_\lambda$ to $\mathcal{S}^{(l,m)}_\lambda$; for such λ, the tensor

$$X = \mathcal{T}^{(l,m)}_\lambda U$$

solves equation (20), for any U.

To find $\mathcal{T}^{(l,m)}_\lambda$, denote

$$U^k = U + \frac{1}{\lambda}(\tau\sigma)^1 U + \frac{1}{\lambda^2}(\tau\sigma)^2 U + \dots + \frac{1}{\lambda^k}(\tau\sigma)^k U,$$

and

$$\begin{aligned}\mu(\lambda) &= (-1)^0\lambda^l\Sigma^l_0 + (-1)^1\lambda^{l-1}\Sigma^l_1 + (-1)^2\lambda^{l-2}\Sigma^l_2\\ &\quad + \dots + (-1)^{l-2}\lambda^2\Sigma^l_{l-2} + (-1)^{l-1}\lambda^1\Sigma^l_{l-1} + (-1)^l\lambda^0\Sigma^l_l.\end{aligned} \tag{22}$$

Lemma 3. (a) *μ can be expressed as*

$$\mu(\lambda) = (\lambda - N_{l-1})(\lambda - N_{l-2})(\lambda - N_{l-3}) \dots (\lambda - N_1)(\lambda - N_0). \tag{23}$$

(b) *At $\lambda = \lambda_0 = (l+1)m$,*

$$\mu(\lambda_0) = \frac{l!\,(l+m)!}{m!}. \tag{24}$$

Proof. (a) The proof is based on the properties (8) of the symmetric polynomials Σ^p_q. We know that

$$\begin{aligned}
&(-1)^0\lambda^p\Sigma^p_0 = (-1)^0\lambda^p\Sigma^{p-1}_0,\\
&(-1)^1\lambda^{p-1}\Sigma^p_1 = (-1)^1\lambda^{p-1}\Sigma^{p.-1}_1 + (-1)^1\lambda^{p-1}\Sigma^{p-1}_0 N_{p-1},\\
&\dots\\
&(-1)^{p-1}\lambda^1\Sigma^p_{p-1} = (-1)^{p-1}\lambda^1\Sigma^{p-1}_{p-1} + (-1)^{p-1}\lambda^1\Sigma^{p-1}_{p-2}N_{p-1},\\
&(-1)^p\Sigma^p_p = (-1)^p\Sigma^{p-1}_{p-1}N_{p-1},
\end{aligned}$$

i.e., after some calculation,

$$\begin{aligned}
&(-1)^0\lambda^p\Sigma^p_0 + (-1)^1\lambda^{p-1}\Sigma^p_1 + (-1)^2\lambda^{p-2}\Sigma^p_2 + (-1)^3\lambda^{p-3}\Sigma^p_3\\
&\quad + \dots + (-1)^{p-2}\lambda^2\Sigma^p_{p-2} + (-1)^{p-1}\lambda^1\Sigma^p_{p-1} + (-1)^p\Sigma^p_p\\
&\quad = (\lambda - N_{p-1})((-1)^0\lambda^{p-1}\Sigma^{p-1}_0 + (-1)^1\lambda^{p-2}\Sigma^{p.-1}_1 + (-1)^2\lambda^{p-3}\Sigma^{p-1}_2\\
&\quad + \dots + (-1)^{p-2}\lambda^1\Sigma^{p-1}_{p-2} + (-1)^{p-1}\lambda^0\Sigma^{p-1}_{p-1}).
\end{aligned}$$

Since this equality holds for all p, we have

$$\begin{aligned}
&(-1)^0\lambda^p\Sigma^p_0 + (-1)^1\lambda^{p-1}\Sigma^p_1 + (-1)^2\lambda^{p-2}\Sigma^p_2 + (-1)^3\lambda^{p-3}\Sigma^p_3\\
&\quad + \dots + (-1)^{p-2}\lambda^2\Sigma^p_{p-2} + (-1)^{p-1}\lambda^1\Sigma^p_{p-1} + (-1)^p\Sigma^p_p\\
&\quad = (\lambda - N_{p-1})((-1)^0\lambda^{p-1}\Sigma^{p-1}_0 + (-1)^1\lambda^{p-2}\Sigma^{p.-1}_1 + (-1)^2\lambda^{p-3}\Sigma^{p-1}_2\\
&\quad + \dots + (-1)^{p-2}\lambda^1\Sigma^{p-1}_{p-2} + (-1)^{p-1}\lambda^0\Sigma^{p-1}_{p-1})\\
&\quad = (\lambda - N_{p-1})(\lambda - N_{p-2})(\lambda - N_{p-2}) \dots (\lambda - N_2)(\lambda - N_1)(\lambda - N_0).
\end{aligned}$$

For $p = l$ we get (23).
(b) (24) follows from Theorem 2. □

Now we are in a position to prove the following result.

Theorem 12. *The operator $\mathcal{S}^{(l,m)}_\lambda$ is invertible if and only if*

$$\lambda \neq (k+1)(m-l+k), \quad k = 0, 1, 2, \dots, l-1, l.$$

In this case $(\lambda - m(l+1))\mu(\lambda) \neq 0$, and the inverse operator $U \longrightarrow \mathcal{T}^{(l,m)}_\lambda U$ is given by

$$\begin{aligned}
\mathcal{T}^{(l,m)}_\lambda U &= \frac{\lambda}{(\lambda - m(l+1))\mu(\lambda)}\sigma^l\tau^l U\\
&\quad + \frac{\lambda}{\mu(\lambda)}\Big((-1)^0\Sigma^l_0\lambda^{l-1}U^{l-1} + (-1)^1\Sigma^l_1\lambda^{l-2}U^{l-2}\\
&\quad + (-1)^2\Sigma^l_2\lambda^{l-3}U^{l-3} + \dots + (-1)^{l-2}\Sigma^l_{l-2}\lambda^1U^1 + (-1)^{l-1}\Sigma^l_{l-1}\lambda^0U^0\Big).
\end{aligned} \tag{25}$$

Proof. 1. Let λ be such that there exists $\mathcal{T}_\lambda^{(l,m)}$. Then for every U, there exists a unique tensor X satisfying (20). We have for these U and X

$$U = X - \frac{1}{\lambda}(\tau\sigma)^1 X,$$
$$\frac{1}{\lambda}(\tau\sigma)^1 U = \frac{1}{\lambda}(\tau\sigma)^1 X - \frac{1}{\lambda^2}(\tau\sigma)^2 X,$$
$$\frac{1}{\lambda^2}(\tau\sigma)^2 U = \frac{1}{\lambda^2}(\tau\sigma)^2 X - \frac{1}{\lambda^3}(\tau\sigma)^3 X,$$
$$\dots$$
$$\frac{1}{\lambda^{k-1}}(\tau\sigma)^{k-1} U = \frac{1}{\lambda^{k-1}}(\tau\sigma)^{k-1} X - \frac{1}{\lambda^k}(\tau\sigma)^k X,$$

hence

$$U + \frac{1}{\lambda}(\tau\sigma)^1 U + \frac{1}{\lambda^2}(\tau\sigma)^2 U + \ldots + \frac{1}{\lambda^{k-1}}(\tau\sigma)^{k-1} U = X - \frac{1}{\lambda^k}(\tau\sigma)^k X,$$

i.e., for all $k \geq 1$,

$$\frac{1}{\lambda^k}(\tau\sigma)^k X = X - U^{k-1}, \tag{26}$$

where

$$U^{k-1} = U + \frac{1}{\lambda}(\tau\sigma)^1 U + \frac{1}{\lambda^2}(\tau\sigma)^2 U + \ldots + \frac{1}{\lambda^{k-1}}(\tau\sigma)^{k-1} U$$

$(U_0 = U)$. On the other hand, by Theorem 1, (10),

$$\begin{aligned}\sigma^l\tau^l X &= (-1)^0\Sigma_0^l(\tau\sigma)^l X + (-1)^1\Sigma_1^l(\tau\sigma)^{l-1} X + (-1)^2\Sigma_2^l(\tau\sigma)^{l-2} X \\ &\quad + \ldots + (-1)^{l-1}\Sigma_{l-1}^l(\tau\sigma)^1 X + (-1)^l\Sigma_l^l X.\end{aligned} \tag{27}$$

We now combine (26) and (27). We obtain

$$(\tau\sigma)^1 X = \lambda^1(X - U^0),$$
$$(\tau\sigma)^2 X = \lambda^2(X - U^1),$$
$$\dots$$
$$(\tau\sigma)^{l-1} X = \lambda^{l-1}(X - U^{l-2}),$$
$$(\tau\sigma)^l X = \lambda^l(X - U^{l-1}),$$

and, with $\mu(\lambda)$ defined by (22),

$$\begin{aligned}\sigma^l\tau^l X &= \mu(\lambda)X - (-1)^0\Sigma_0^l\lambda^l U^{l-1} - (-1)^1\Sigma_1^l\lambda^{l-1}U^{l-2} \\ &\quad - (-1)^2\Sigma_2^l\lambda^{l-2}U^{l-3} - \ldots - (-1)^{l-2}\Sigma_{l-2}^l\lambda^2 U^1 - (-1)^{l-1}\Sigma_{l-1}^l\lambda^1 U^0.\end{aligned} \tag{28}$$

We find $\sigma^l\tau^l X$ from equation (20), and then compute X from equation (28). From Lemma 2, (a) we know that $\tau\sigma\sigma^l\tau^l X = \sigma^l\tau^l\tau\sigma X = m(l+1)\sigma^l\tau^l X$. Thus, we have from (20)

$$\begin{aligned}\sigma^l\tau^l U &= \sigma^l\tau^l X - \frac{1}{\lambda}\sigma^l\tau^l\tau\sigma X \\ &= \sigma^l\tau^l X - \frac{m(l+1)}{\lambda}\sigma^l\tau^l X = \frac{\lambda - m(l+1)}{\lambda}\sigma^l\tau^l X.\end{aligned} \tag{29}$$

Substituting from (29) to (28) we obtain

$$\frac{\lambda - m(l+1)}{\lambda}\sigma^l\tau^l X = \frac{\lambda - m(l+1)}{\lambda}\mu(\lambda)X$$
$$- \frac{\lambda - m(l+1)}{\lambda}\Big((-1)^0\Sigma_0^l\lambda^l U^{l-1} + (-1)^1\Sigma_1^l\lambda^{l-1}U^{l-2}$$
$$+ (-1)^2\Sigma_2^l\lambda^{l-2}U^{l-3} + \ldots + (-1)^{l-2}\Sigma_{l-2}^l\lambda^2 U^1 + (-1)^{l-1}\Sigma_{l-1}^l\lambda^1 U^0\Big)$$
$$= \sigma^l\tau^l U,$$

that is,

$$\frac{(\lambda - m(l+1))\mu(\lambda)}{\lambda}X = \sigma^l\tau^l U + \frac{\lambda - m(l+1)}{\lambda}\Big((-1)^0\Sigma_0^l\lambda^l U^{l-1}$$
(30) $$+ (-1)^1\Sigma_1^l\lambda^{l-1}U^{l-2} + (-1)^2\Sigma_2^l\lambda^{l-2}U^{l-3} + \ldots + (-1)^{l-2}\Sigma_{l-2}^l\lambda^2 U^1$$
$$+ (-1)^{l-1}\Sigma_{l-1}^l\lambda^1 U^0\Big).$$

But by hypothesis, to the given U we have a unique tensor X satisfying (20); thus, λmust differ from the roots of the polynomial $(\lambda - m(l+1))\mu(\lambda)$.

From Lemma3 we now see that (30) gives formula (25) of Theorem3.

2. The converse can be proved by reversing our arguments. □

Remark 6. If we denote the polynomial (22) by μ^l, then by Lemma 3, $\mu^{l+1}(\lambda) = (\lambda - m(l+1))\mu^l(\lambda)$, i.e.,

$$\mu^{l+1}(\lambda) = (\lambda - N_l)(\lambda - N_{l-1})(\lambda - N_{l-2})(\lambda - N_{l-3})\ldots(\lambda - N_1)(\lambda - N_0).$$

Thus, Theorem 3 says that the operator $\mathcal{S}_\lambda^{(l,m)}$ is invertible if and only if λ differs from the roots of the polynomial μ^{l+1}.

7. Multisymmetric-symmetric tensors

In Section 5, we obtained the trace decomposition equations for tensors $A \in \mathrm{Z}_s^r E$ by means of two operators, q and tr. Recall that for a given tensor A, the problem is to find traceless tensors $X \in \mathrm{Z}_{s-1}^{r-1}E$ and $U \in \mathrm{Z}_s^r E$ such that

(1) $$A = \mathrm{q}\,X + U.$$

A solution (X, U) can be easily found by means of the commutation formula for q and tr: we have $\mathrm{tr}\,A = \mathrm{tr}\,\mathrm{q}\,X + \mathrm{tr}\,U = X - \mathrm{q}\,\mathrm{tr}\,X + \mathrm{tr}\,U = X$, because by hypothesis, $\mathrm{tr}\,X = 0$ and $\mathrm{tr}\,U = 0$; then $U = A - \mathrm{q}\,X$.

Now we wish to formulate the *complete trace decomposition equations* for tensor spaces $A \in \mathrm{Z}_s^{(r_1,r_2,\ldots,r_p)}E$, where $2 \le p \le s$. To this purpose we need analogues of the operators q and tr, applicable to several multi-indices; we also need the elementary symmetrization operators introduced in Section 6, which could be applied to any pair of the multi-indices.

Thus, we define the operators tr_α and q_α by means of formulas (1), (2), Section 5, applied to the multi-index $J_\alpha = (j_1 j_2 \ldots j_{r_\alpha})$. For any two multi-indices $J_\alpha = (j_1 j_2 \ldots j_{r_\alpha})$ and $J_\beta = (k_1 k_2 \ldots k_{r_\beta})$, we define the elementary symmetrization operators $\sigma_{\alpha,\beta}$ by means of formula (3), Section 6.

If, for example, $\alpha = 1$, $r_\alpha = l$, $\beta = 2$, $r_\beta = m$, and U belongs to the tensor space $\mathrm{Z}_s^{(l,m)}E$, $U = U^{j_1 j_2 \dots j_l \; k_1 k_2 \dots k_m}{}_{i_1 i_2 \dots i_s}$, we have

$$\mathrm{q}_1 U = C_{l,s} \delta^{j_1}_{i_1} U^{j_2 j_3 \dots j_{l+1} \; k_1 k_2 \dots k_m}{}_{i_2 i_3 \dots i_{s+1}} \,\mathrm{alt}\,(i_1 i_2 \dots i_{s+1})\,\mathrm{sym}\,(j_1 j_2 \dots j_{l+1}),$$
$$\mathrm{q}_2 U = C_{m,s} \delta^{k_1}_{i_1} U^{j_1 j_2 \dots j_l \; k_2 k_3 \dots k_{m+1}}{}_{i_2 i_3 \dots i_{s+1}} \,\mathrm{alt}\,(i_1 i_2 \dots i_{s+1})\,\mathrm{sym}\,(k_1 k_2 \dots k_{m+1}),$$
$$\mathrm{tr}_1 U = U^{p j_1 j_2 \dots j_{l-1} \; k_1 k_2 \dots k_m}{}_{p i_1 i_2 \dots i_{s-1}},$$
$$\mathrm{tr}_2 U = U^{j_1 j_2 \dots j_l \; p k_1 k_2 \dots k_{m-1}}{}_{p i_1 i_2 \dots i_{s-1}},$$
$$\sigma_{1,2} U = (l{+}1) U^{j_2 j_3 \dots j_l j_{l+1} \; j_1 k_1 k_2 \dots k_{m-1}}{}_{i_1 i_2 \dots i_s} \quad \mathrm{sym}\,(j_1 j_2 \dots j_l j_{l+1}),$$
$$\sigma_{2,1} U = (m{+}1) U^{k_1 j_1 j_2 \dots j_{l-1} \; k_2 k_3 \dots k_m k_{m+1}}{}_{i_1 i_2 \dots i_s} \quad \mathrm{sym}\,(k_1 k_2 \dots k_m k_{m+1}).$$

Using these definitions, we can prove the following two theorems. Denote

$$\sigma = \sigma_{1,2} = (l+1)\mathrm{sym}_{1,2}, \qquad \tau = \sigma_{2,1} = (m+1)\mathrm{sym}_{2,1}.$$

Theorem 13. *For $i = 1, 2$,*

$$\mathrm{q}_i \mathrm{q}_i U = 0, \quad \mathrm{tr}_i \mathrm{tr}_i U = 0, \quad U = \mathrm{q}_i \mathrm{tr}_i U + \mathrm{tr}_i \mathrm{q}_i U,$$
$$q_2 q_1 U = -\frac{n+l-s-1}{n+m-s-1} \frac{n+m-s}{n+l-s} q_1 q_2 U, \quad \mathrm{tr}_1 \mathrm{tr}_2 U = -\mathrm{tr}_2 \mathrm{tr}_1 U.$$

Proof. See Section 5, Theorem 1. □

Theorem 14. *For any $U \in \mathrm{Z}_s^{(l,m)}E$, $U = U^{j_1 j_2 \dots j_l \; k_1 k_2 \dots k_m}_{i_1 i_2 \dots i_s}$,*

$$\mathrm{tr}_2 \mathrm{q}_1 U = \frac{1}{n+l-s} \sigma U - \frac{n+l-s+1}{n+l-s} \mathrm{q}_1 \mathrm{tr}_2 U,$$
$$\mathrm{tr}_1 \mathrm{q}_2 U = \frac{1}{n+m-s} \tau U - \frac{n+m-s+1}{n+m-s} \mathrm{q}_2 \mathrm{tr}_1 U,$$
$$\mathrm{tr}_2 \tau U = \mathrm{tr}_1 U + \tau \mathrm{tr}_2 U,$$
$$\mathrm{tr}_1 \sigma U = \mathrm{tr}_2 U + \sigma \mathrm{tr}_1 U,$$
$$\mathrm{tr}_1 \tau U = \tau \mathrm{tr}_1 U,$$
$$\mathrm{tr}_2 \sigma U = \sigma \mathrm{tr}_2 U,$$
$$\tau \mathrm{q}_1 U = \frac{n+m-s}{n+l-s} \mathrm{q}_2 U + \frac{n+l-1-s}{n+l-s} \mathrm{q}_1 \tau U,$$
$$\sigma \mathrm{q}_2 U = \frac{n+l-s}{n+m-s} \mathrm{q}_1 U + \frac{n+m-1-s}{n+m-s} \mathrm{q}_2 \sigma U,$$
$$\tau \mathrm{q}_1 U = \frac{n+l+1-s}{n+l-s} \mathrm{q}_1 \tau U,$$
$$\tau \mathrm{q}_2 U = \frac{n+m+1-s}{n+m-s} \mathrm{q}_2 \tau U,$$
$$\mathrm{q}_2 \mathrm{tr}_1 \mathrm{q}_1 \mathrm{tr}_2 U = \frac{1}{(n+m-s)(n+l-s+1)} \tau \sigma U - \frac{n+l-s}{(n+m-s)(n+l-s+1)} \tau \mathrm{tr}_2 \mathrm{q}_1 U$$

$$+\frac{n+l-s}{n+l-s+1}\mathrm{tr}_1\mathrm{q}_1(U-\mathrm{tr}_2\mathrm{q}_2U),$$

$$\mathrm{tr}_2\mathrm{q}_1\mathrm{tr}_1\mathrm{q}_2U$$

$$=\frac{1}{(n+l-1-s)(n+m-s)}(\sigma\tau U-(n+m-s+1)\sigma\mathrm{q}_2\mathrm{tr}_1U)$$

$$-\frac{(n+l-s)}{(n+l-1-s)(n+m-s)}(\mathrm{q}_1\mathrm{tr}_2\tau U-(n+m-s+1)\mathrm{q}_1\mathrm{tr}_2\mathrm{q}_2\mathrm{tr}_1U),$$

$$\tau\sigma\mathrm{tr}_1\mathrm{tr}_2U=\mathrm{tr}_1\mathrm{tr}_2\tau\sigma U,$$

$$\sigma\mathrm{tr}_1\mathrm{tr}_2U=\mathrm{tr}_1\mathrm{tr}_2\sigma U,$$

$$\tau\mathrm{tr}_1\mathrm{tr}_2U=\mathrm{tr}_1\mathrm{tr}_2\tau U.$$

Proof. The formulas can be obtained by a direct computation. □

From Theorem 1 we can now easily deduce the general structure of the trace decomposition equations for tensors $A\in \mathrm{Z}_s^{(r_1,r_2,\ldots,r_p)}E$, generalizing equation (1). If $p=2$, then the trace decomposition equation for a tensor $A\in \mathrm{Z}_s^{(l,m)}E$ is

(2) $$A=\mathrm{q}_2\mathrm{q}_1X+\mathrm{q}_1U+\mathrm{q}_2V+Z,$$

where X, U, V, Z are unknown traceless tensors. If $p=3$ and $A\in \mathrm{Z}_s^{(l,m,p)}E$, the trace decomposition equation is

(3) $$A=\mathrm{q}_3\mathrm{q}_2\mathrm{q}_1X+\mathrm{q}_2\mathrm{q}_1U_1+\mathrm{q}_3\mathrm{q}_1U_2+\mathrm{q}_3\mathrm{q}_2U_3+\mathrm{q}_1V_1+\mathrm{q}_2V_2+\mathrm{q}_3V_3+Z,$$

where $X,U_1,U_2,U_3,V_1,V_2,V_3,Z$ are unknown traceless tensors. The general trace decomposition equation can be formulated in the same way; then Theorem2 gives us the method for solving this equation.

8. The trace decomposition of tensors of characteristic $(l,m;s)$

Let A be a tensor of characteristic $(l,m;s)$. The *trace decomposition problem* for$A$ is the problem of solvability of the trace decomposition equation

(1) $$A=\mathrm{q}_2\mathrm{q}_1X+\mathrm{q}_1U+\mathrm{q}_2V+Z$$

for unknown *traceless* tensors X, U, V, Z (Section 7). Our main result in this section is the following explicit *trace decomposition formula.*

Theorem 15. *For any tensor $A\in \mathrm{Z}_s^{(l,m)}E$, the tensors*

$$X=\mathcal{T}_{\lambda_0}^{(l-1,m-1)}\mathrm{tr}_1\mathrm{tr}_2A,$$

$$U=\mathcal{T}_{\lambda_1}^{(l-1,m)}\mathrm{tr}_1(A-\mathrm{q}_2\mathrm{q}_1X)-\frac{1}{n-s+m}\mathcal{T}_{\lambda_1}^{(l-1,m)}\tau\mathrm{tr}_2(A-\mathrm{q}_2\mathrm{q}_1X),$$

$$V=\mathcal{T}_{\lambda_1}^{(l,m-1)}\mathrm{tr}_2(A-\mathrm{q}_2\mathrm{q}_1X)-\frac{1}{n-s+l}\mathcal{T}_{\lambda_1}^{(l,m-1)}\sigma\mathrm{tr}_1(A-\mathrm{q}_2\mathrm{q}_1X),$$

$$Z=A-\mathrm{q}_1U-\mathrm{q}_2V-\mathrm{q}_2\mathrm{q}_1X$$

solve the trace decomposition equation (1)*.*

Proof. In the following computations, we use formulas given in Theorem 1 and Theorem 2, Section 7.

Applying the operator tr_2 to (1) we have $\mathrm{tr}_2 A = \mathrm{q}_1 X - \mathrm{q}_2 \mathrm{tr}_2 \mathrm{q}_1 X + \mathrm{tr}_2 \mathrm{q}_1 U + V$, and $\mathrm{tr}_1 \mathrm{tr}_2 A = X - \mathrm{tr}_1 \mathrm{q}_2 \mathrm{tr}_2 \mathrm{q}_1 X$. We compute the second term, $\mathrm{tr}_1 \mathrm{q}_2 \mathrm{tr}_2 \mathrm{q}_1 X$. Since $char X = (l-1, m-1; s-2)$ and X is traceless, we have

$$\mathrm{tr}_2 \mathrm{q}_1 X = \frac{1}{n+l-1-s+2} \sigma X - \frac{n+l-1-s+2+1}{n+l-1-s+2} \mathrm{q}_1 \mathrm{tr}_2 X = \frac{1}{n+l-s+1} \sigma X.$$

Similarly, setting $V = \mathrm{tr}_2 \mathrm{q}_1 X$, we have $char V = (l, m-2; s-2)$. Thus,

(2) $$\mathrm{tr}_1 q_2 \mathrm{tr}_2 \mathrm{q}_1 X = \frac{1}{n+m-s} \frac{1}{n+l-s+1} \tau \sigma X.$$

Denoting $\lambda_0 = (n+m-s)(n+l-s+1)$ we see that X must satisfy the equation

$$\mathrm{tr}_1 \mathrm{tr}_2 A = X - \frac{1}{\lambda_0} \tau \sigma X,$$

or, in the notation of Section 6, (21),

$$\mathcal{S}_{\lambda_0}^{(l-1,m-1)} X = \mathrm{tr}_1 \mathrm{tr}_2 A.$$

We know that for existence of a solution it is necessary and sufficient that $\lambda_0 \neq (k+1)(m-l+k)$ for every $k = 0, 1, 2, \ldots, l-2, l-1$. This condition is obviously satisfied: two solutions of the equation $(n+m-s)(n+l-s+1) = (k+1)(m-l+k)$, $k_1 = n-s+l$ and $k_2 = -n+s-m-1$, satisfy $k_1 > l-1$ and $k_2 = -n+s-m-1 < 0$. Consequently,

$$X = \mathcal{T}_{\lambda_0}^{(l-1,m-1)} \mathrm{tr}_1 \mathrm{tr}_2 A.$$

The trace decomposition equation (1) is thus reduced to the equation

$$A - \mathrm{q}_2 \mathrm{q}_1 \mathcal{T}_{\lambda_0}^{(l-1,m-1)} \mathrm{tr}_1 \mathrm{tr}_2 A = \mathrm{q}_1 U + \mathrm{q}_2 V + Z.$$

Denote

(3) $$A' = A - \mathrm{q}_2 \mathrm{q}_1 \mathcal{T}_{\lambda_0}^{(l-1,m-1)} \mathrm{tr}_1 \mathrm{tr}_2 A.$$

Then we have

$$\mathrm{tr}_1 A' = \mathrm{tr}_1 \mathrm{q}_1 U + \mathrm{tr}_1 \mathrm{q}_2 V + \mathrm{tr}_1 Z = U + \mathrm{tr}_1 \mathrm{q}_2 V,$$
$$\mathrm{tr}_2 A' = \mathrm{tr}_2 \mathrm{q}_1 U + \mathrm{tr}_2 \mathrm{q}_2 V + \mathrm{tr}_2 Z = \mathrm{tr}_2 \mathrm{q}_1 U + V.$$

But $char U = (l-1, m, s-1)$ and $char V = (l, m-1, s-1)$, so that

$$\mathrm{tr}_2 \mathrm{q}_1 U = \frac{1}{n+l-s} \sigma U, \quad \mathrm{tr}_1 \mathrm{q}_2 V = \frac{1}{n+m-s} \tau V,$$

hence

(4) $$\mathrm{tr}_1 A' = U + \frac{1}{n+m-s} \tau V, \quad \mathrm{tr}_2 A' = \frac{1}{n+l-s} \sigma U + V.$$

Applying symmetrizations, we have

$$\sigma \mathrm{tr}_1 A' = \sigma U + \frac{1}{n+m-s} \sigma \tau V, \quad \tau \mathrm{tr}_2 A' = \frac{1}{n+l-s} \tau \sigma U + V.$$

Then

$$\frac{1}{n+l-s}\sigma \mathrm{tr}_1 A' = \frac{1}{n+l-s}\sigma U + \frac{1}{n+l-s}\frac{1}{n+m-s}\sigma\tau V,$$
$$\frac{1}{n+m-s}\tau \mathrm{tr}_2 A' = \frac{1}{n+m-s}\frac{1}{n+l-s}\tau\sigma U + \frac{1}{n+m-s}\tau V.$$

From (4)

$$\frac{1}{n+l-s}\sigma \mathrm{tr}_1 A' = \mathrm{tr}_2 A' - V + \frac{1}{n+l-s}\frac{1}{n+m-s}\sigma\tau V,$$
$$\frac{1}{n+m-s}\tau \mathrm{tr}_2 A' = \frac{1}{n+m-s}\frac{1}{n+l-s}\tau\sigma U + \mathrm{tr}_1 A' - U,$$

Denoting $\lambda_1 = (n-s+l)(n-s+m)$ we obtain the equations

$$\begin{aligned} &\mathrm{tr}_2 A' - \frac{1}{n+l-s}\sigma \mathrm{tr}_1 A' = V - \frac{1}{\lambda_1}\sigma\tau V, \\ &\mathrm{tr}_1 A' - \frac{1}{n+m-s}\tau \mathrm{tr}_2 A' = U - \frac{1}{\lambda_1}\tau\sigma U. \end{aligned} \tag{5}$$

We can apply Theorem 3 of Section 6 to each of these equations.

We have to analyze solvability. Note that in (5), $charV = (l, m-1, s-1)$; thus, for existence of a solution V it is necessary and sufficient that λ_1 be different from $k = 0, 1, 2, \ldots, l-1, l$. Namely, we want to show that the solutions x of the equation

$$(n-s+l)(n-s+m) = (x+1)(m-1-l+x) \tag{6}$$

satisfy $x \neq k$. Suppose that (6) has a (real) solution. Then the quadratic form

$$x^2 + (m-l)x + m - 1 + l - (n-s+l)(n-s+m) \tag{7}$$

has non-negative discriminant $D = (m-l-2)^2 + 4(n-s+l)(n-s+m) \geq 0$. D can be viewed as a non-negative quadratic form in $n-s$,

$$D = 4(n-s)^2 + 4(n-s)(m+l) + 4lm + (m-l-2)^2 \geq 0. \tag{8}$$

Computing its discriminant, we have $D' = 64(m-l-1)$ so (8) implies $m-l-1 = 0$.

In this case the double root of D is $-(m+l)/2$, and we have

$$D = 4\left(n - s + \frac{m+l}{2}\right)^2 = (2(n-s) + m + l)^2.$$

The roots of the quadratic form (7) are given by $x_1 = n-s+l$, $x_2 = -(n-s)-m$. Obviously $x_{1,2} \neq 0, 1, 2, \ldots, l-1, l$ as desired.

An analogous result is true for the second equation (5). Writing (5) in the form

$$\mathcal{S}^{(l,m-1)}_{\lambda_1} V = \mathrm{tr}_2 A' - \frac{1}{n+l-s}\sigma \mathrm{tr}_1 A',$$
$$\mathcal{S}^{(l,m-1)}_{\lambda_1} U = \mathrm{tr}_1 A' - \frac{1}{n+m-s}\tau \mathrm{tr}_2 A',$$

we get the solution

$$V = \mathcal{T}_{\lambda_1}^{(l,m-1)} \left(\mathrm{tr}\,_2 A' - \frac{1}{n-s+l} \sigma \mathrm{tr}\,_1 A' \right),$$
$$U = \mathcal{T}_{\lambda_1}^{(l-1,m)} \left(\mathrm{tr}\,_1 A' - \frac{1}{n-s+m} \tau \mathrm{tr}\,_2 A' \right),$$

where A' is given by (3).

Finally, Z is determined from equation (1). Summarizing our results, we have

$$Z = A - \mathrm{q}\,_1 U - \mathrm{q}\,_2 V - \mathrm{q}\,_2 \mathrm{q}\,_1 X,$$
$$X = \mathcal{T}_{\lambda_0}^{(l-1,m-1)} \mathrm{tr}\,_1 \mathrm{tr}\,_2 A,$$
$$V = \mathcal{T}_{\lambda_1}^{(l,m-1)} \left(\mathrm{tr}\,_2 A' - \frac{1}{n-s+l} \sigma \mathrm{tr}\,_1 A' \right),$$
$$U = \mathcal{T}_{\lambda_1}^{(l-1,m)} \left(\mathrm{tr}\,_1 A' - \frac{1}{n-s+m} \tau \mathrm{tr}\,_2 A' \right),$$

where $A' = A - \mathrm{q}\,_2 \mathrm{q}\,_1 \mathcal{T}_{\lambda_0}^{(l-1,m-1)} \mathrm{tr}\,_1 \mathrm{tr}\,_2 A$ is given by (3). This completes the proof. □

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