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Department of Algebra and Geometry, Palacky University
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Variational principles on the frame bundles

Ján Brajerčík and Demeter Krupka[1]

Abstract. Variational principles on the r-jet prolongations of the frame bundles, invariant with respect to the structure group are discussed. We establish basic properties of some Lepage equivalents and Euler-Lagrange forms of invariant lagrangians. The corresponding Noether's theorem and its consequences for the Euler-Lagrange equations are also considered.

Keywords. frame bundle, variational principle, lagrangian, Noether's invariance.
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1. Introduction

Let $\mu : FX \longrightarrow X$ be the frame bundle over an n-dimensional manifold X and let J^rFX be its r-jet prolongation. We consider the standard right action of the general linear group $Gl_n(\mathbb{R})$ on FX and the canonical prolongation of the action of $Gl_n(\mathbb{R})$ to J^rFX.

The goal of this paper is twofold. First, we describe lagrangians on J^rFX which are invariant with respect to the action of $Gl_n(\mathbb{R})$. We give explicit characterization of other basic concepts of the variational theory, such as Euler-Lagrange forms and Lepage equivalents of invariant lagrangians on J^1FX and J^2FX. We work with charts adapted to the action of $Gl_n(\mathbb{R})$ on respective spaces. For underlying spaces and basic notions of the variational theory on fibered spaces we refer to [4], [6], [9], [7], [14]. Some notions related to the frame bundles and invariance can be found in [3], [5], [8], [10], [11].

Secondly, we apply the well-known Noether's theorem to invariant lagrangians on J^1FX and J^2FX. The fundamental vector fields of the right action of $Gl_n(\mathbb{R})$ are the generators of invariant transformations; then Noether's theorem gives us the so called Noether's currents for every from n^2 linearly independent fundamental vector fields. The Noether's currents are, in well-known sense, invariants along any extremal of the given variational problem. Our definitions corresponds with [6], [9], [15]. We study the problem as to how the Noether's currents can be used to simplify the Euler-Lagrange equations. We show that in case of invariant lagrangian defined on J^1FX, the set of n^2 second order Euler-Lagrange equations is reduced to the same number of equations of the first order.

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Note that for variational problems on principal fiber bundles there are several different concepts of invariance. Muñoz and Rosado [12], [13] consider invariance with respect to diffeomorphisms of the base manifold, or, which is the same, natural variational problems (of the first order) in the sense of Krupka [7]. Castrillón, Muñoz and Ratiu focus on invariance of the first order lagrangians in a pricipal fiber bundle P with respect to the prolongation of the structure group to J^1P [1], [2].

2. Lagrange structures

2.1. Jet prolongations of fibered manifolds

We introduce our notation. Y is a *fibered manifold* with oriented *base manifold* X and *projection* π. We denote $n = \dim X$, $m = \dim Y - n$. J^rY is *r-jet prolongation* of Y. If $J^r_x\gamma \in J^rY$, where γ is a smooth section of Y defined at $x \in X$, the *canonical jet projections* $\pi^{r,s} : J^rY \longrightarrow J^sY$, $0 \leq s \leq r$, $\pi^{r,0} : J^rY \longrightarrow Y$ (the target projection), $\pi^r : J^rY \longrightarrow X$ (the source projection), are defined by $\pi^{r,s}(J^r_x\gamma) = J^s_x\gamma$, $\pi^{r,0}(J^r_x\gamma) = \gamma(x)$, and $\pi^r(J^r_x\gamma) = x$, respectively.

2.2. The structure of forms on jet spaces

If $W \subset Y$ is an open set, we denote by $\Omega^r_0 W$ the ring of functions on $W^r = (\pi^{r,0})^{-1}(W)$. The $\Omega^r_0 W$-module of differential q-forms on W^r is denoted by $\Omega^r_q W$, and the exterior algebra of forms on W^r is denoted by $\Omega^r W$. $\Omega^r_{q,Y} W$ (resp. $\Omega^r_{q,X} W$) denotes the module of $\pi^{r,0}$-*horizontal* (resp. π^r-*horizontal*) q-forms.

Let $\eta \in \Omega^r_q W$ be a form. There is a unique decomposition

$$(\pi^{r+1,r})^*\eta = h\eta + p_1\eta + p_2\eta + \ldots + p_q\eta$$

of η into its *horizontal*, or 0-contact, component $h\eta = p_0\eta$ and the *k-contact* components $p_k\eta$, $1 \leq k \leq q$. The mapping $\Omega^r_q W \ni \eta \longrightarrow h\eta \in \Omega^{r+1}_{q,X} W$ is called the π-*horizontalization*.

2.3. Lagrangians

A *lagrangian* for Y is any form $\lambda \in \Omega^r_{n,X} W$. The number r is called the *order* of λ. In a fibered chart (V^r, ψ^r), $\psi^r = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \ldots, y^\sigma_{j_1 j_2 \ldots j_r})$, $1 \leq i \leq n$, $1 \leq \sigma \leq m$, $1 \leq j_1, j_2, \ldots, j_r \leq n$, on J^rY, associated with a fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, $V \subset W$, on Y, a lagrangian of order r defined on V^r has an expression $\lambda = L\omega_0$, where $L : V^r \longrightarrow \mathbb{R}$ is the *Lagrange function* and $\omega_0 = dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$. We also denote $\omega_j = i_{\partial/\partial x^j}\omega_0 = (-1)^{j-1} dx^1 \wedge dx^2 \wedge \ldots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \ldots \wedge dx^n$, and $\omega^\sigma_{j_1 j_2 \ldots j_k} = dy^\sigma_{j_1 j_2 \ldots j_k} - y^\sigma_{j_1 j_2 \ldots j_k l} dx^l$, where $1 \leq j, j_1, j_2, \ldots, j_k \leq n$, $1 \leq \sigma \leq m$, $0 \leq k \leq r$. Given a lagrangian λ, the pair (π, λ) is called a *Lagrange structure*.

2.4. Lepage forms, the Euler-Lagrange form

A differential form $\rho \in \Omega^s_n W$ is *Lepage*, if $p_1 d\rho \in \Omega^{s+1}_{n+1,Y} W$. If, for some lagrangian $\lambda \in \Omega^r_{n,X} W$ and the Lepage form $\rho \in \Omega^s_n W$, $h\rho = \lambda$ holds, then ρ is said to be a *Lepage equivalent* of a lagrangian λ.

If $\rho \in \Omega_n^s W$ is a Lepage equivalent of a lagrangian $\lambda \in \Omega_{n,X}^r W$, $\lambda = L\omega_0$, then $p_1 d\rho = E_\sigma(L)\omega^\sigma \wedge \omega_0$, where

$$E_\sigma(L) = \sum_{l=0}^{r} (-1)^l d_{p_1} d_{p_2} \dots d_{p_l} \frac{\partial L}{\partial y^\sigma_{p_1 p_2 \dots p_l}}. \tag{1}$$

The form $p_1 d\rho$ is called the *Euler-Lagrange* (E-L) *form* associated with λ, and is usually denoted by E_λ; obviously $E_\lambda \in \Omega^{2r}_{n+1,Y} W$. The components $E_\sigma(L)$ of E_λ are the *Euler-Lagrange expressions.*

Each lagrangian of order r has the Lepage equivalent of order $2r-1$ (see [9]). Each n-form $\rho \in \Omega_n^{r-1} W$ defines a lagrangian of order r on Y, namely the lagrangian $\lambda = h\rho$, the horizontal component of ρ. This lagrangian is said to be *associated* with ρ.

Each lagrangian $\lambda \in \Omega^1_{n,X} W$ has a unique Lepage equivalent $\Theta_\lambda \in \Omega^1_{n,Y} W$ whose order of contactness is ≤ 1. This form is also known as the *Poincaré-Cartan* (P-C) *form* (see [4]). If, in a fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, $V \subset W$, $\lambda = L\omega_0$, then

$$\Theta_\lambda = L\omega_0 + \frac{\partial L}{\partial y^\sigma_i} \omega^\sigma \wedge \omega_i. \tag{2}$$

Note that for $\lambda \in \Omega^1_{n,X} W$ we have another Lepage equivalent, expressed in the fibered chart by

$$\Phi_\lambda = \sum_{k=0}^{n} \frac{1}{k!\,(n-k)!} \frac{1}{k!} \frac{\partial^k L}{\partial y^{\sigma_1}_{j_1} \partial y^{\sigma_2}_{j_2} \dots \partial y^{\sigma_k}_{j_k}} \epsilon_{j_1 j_2 \dots j_k i_{k+1} i_{k+2} \dots i_n} \cdot \omega^{\sigma_1} \wedge \omega^{\sigma_2} \wedge \dots \wedge \omega^{\sigma_k} \wedge dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \dots \wedge dx^{i_n}. \tag{3}$$

Evidently, $\Phi_\lambda \in \Omega^1_n W$. Φ_λ is called the *fundamental* Lepage equivalent of the lagrangian λ.

If $\lambda \in \Omega^2_{n,X} W$, there exists a Lepage equivalent $\Theta_\lambda \in \Omega^3_n W$ of λ. If λ is expressed in some fibered chart by $\lambda = L\omega_0$, then

$$\Theta_\lambda = L\omega_0 + \left(\frac{\partial L}{\partial y^\sigma_i} - d_p \frac{\partial L}{\partial y^\sigma_{pi}} \right) \omega^\sigma \wedge \omega_i + \frac{\partial L}{\partial y^\sigma_{ji}} \omega^\sigma_j \wedge \omega_i \tag{4}$$

(for Φ_λ and Θ_λ see [9]).

2.5. Variations

In this subsection, we introduce the notions allowing us to give a proper formulation of the first variation formula.

Denote by $\Gamma_{\Omega,W}(\pi)$ the set of smooth sections γ of π over compact, n-dimensional submanifold Ω of X with boundary, such that $\gamma(\Omega) \subset W$. Then we have a real-valued function

$$\Gamma_{\Omega,W}(\pi) \ni \gamma \longrightarrow \lambda_\Omega(\gamma) = \int_\Omega J^r \gamma^* \lambda \in \mathbb{R}, \tag{5}$$

called the *variational function* associated with λ over Ω. Let $U \subset X$ be an open set and let $\gamma : U \longrightarrow Y$ be a section. Let ξ be a π-projectable vector field on an open set $W \subset Y$ such that $\gamma(U) \subset W$. If α_t is the local 1-parameter group of ξ and $\alpha_{(0)t}$ its projection, then $\gamma_t = \alpha_t \gamma \alpha^{-1}_{(0)t}$ is a 1-parameter family of sections of Y, depending smoothly on the parameter t. γ_t is *variation* of γ induced by ξ. For variation γ_t of a smooth section

$\gamma \in \Gamma_{\Omega,W}(\pi)$ we have a real-valued function, defined on a neighborhood of point $0 \in \mathbb{R}$,

$$(-\epsilon,\epsilon) \ni t \longrightarrow \lambda_{\alpha_{(0)t}(\Omega)}(\alpha_t \gamma \alpha_{(0)t}^{-1}) = \int\limits_{\alpha_{(0)t}(\Omega)} (J^r(\alpha_t \gamma \alpha_{(0)t}^{-1}))^* \lambda \in \mathbb{R}.$$

Differentiating at $t = 0$ one obtains the number

$$\left\{ \frac{d}{dt} \lambda_{\alpha_{(0)t}(\Omega)}(\alpha_t \gamma \alpha_{(0)t}^{-1}) \right\}_0 = \int\limits_{\Omega} J^r\gamma^* \partial_{J^r\xi}\lambda,$$

where operator $\partial_{J^r\xi}$ denotes Lie derivative by vector field $J^r\xi$. This number is called the *variation of the variational function* λ_Ω at γ, induced by the vector field ξ. According to notation in (5) we can write

$$\int\limits_{\Omega} J^r\gamma^* \partial_{J^r\xi}\lambda = (\partial_{J^r\xi}\lambda)_\Omega(\gamma).$$

2.6. Variational derivatives

The function

(6) $$\Gamma_{\Omega,W}(\pi) \ni \gamma \longrightarrow (\partial_{J^r\xi}\lambda)_\Omega(\gamma) \in \mathbb{R}$$

is the variational function (over Ω) associated with the lagrangian $\partial_{J^r\xi}\lambda$. Function (6) is called the *variational derivative* or the *first variation* of the variational function λ_Ω by the vector field ξ.

If $\rho \in \Omega_n^s W$ is a Lepage equivalent of $\lambda \in \Omega_{n,X}^r W$, ξ a π-projectable vector field on W, then the Lie derivative $\partial_{J^r\xi}\lambda$ can be expressed by the formula

(7) $$\partial_{J^r\xi}\lambda = hi_{J^s\xi}d\rho + hdi_{J^s\xi}\rho.$$

Thus, for any section γ of Y

(8) $$J^r\gamma^*\partial_{J^r\xi}\lambda = J^s\gamma^* i_{J^s\xi}d\rho + dJ^s\gamma^* i_{J^s\xi}\rho.$$

If $\Omega \subset X$ is an n-dimensional submanifold with boundary $\partial\Omega$, then for any section γ of Y defined on Ω

(9) $$\int\limits_{\Omega} J^r\gamma^*\partial_{J^r\xi}\lambda = \int\limits_{\Omega} J^s\gamma^* i_{J^s\xi}d\rho + \int\limits_{\partial\Omega} J^s\gamma^* i_{J^s\xi}\rho.$$

Formulas (7) and (8) are called the *infinitesimal first variation formulas*, and (9) is the *integral first variation formula*.

We say, that a section $\gamma_0 \in \Gamma_{\Omega,W}(\pi)$ is an *extremal* of the variational function (5), or an extremal of λ on Ω, if

$$\int\limits_{\Omega} J^r\gamma_0^*\partial_{J^r\xi}\lambda = 0$$

for all π-projectable vector fields ξ such that $supp(\xi \circ \gamma_0) \subset \Omega$. γ_0 is an extremal of the Lagrange structure (π,λ), or simply an *extremal*, if it is an extremal of the variational function (5) for every Ω in the domain of definition of γ_0.

Let $\lambda \in \Omega_{n,X}^r W$ be a lagrangian, E_λ the Euler-Lagrange form associated with λ, $\gamma : U \longrightarrow Y$ a section of Y, $\Omega \subset U$ a compact n-dimensional submanifold with boundary, and ρ a Lepage equivalent of λ. The following conditions are equivalent:

(a) γ is an extremal on Ω.

(b) For every π-vertical vector field ξ defined on a neighborhood of $\gamma(U)$, such that $supp(\xi \circ \gamma) \subset \Omega$,

(10) $$J^s\gamma^* i_{J^s\xi} d\rho = 0.$$

(c) For any fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, γ satisfies the system of partial differential equations

(11) $$E_\sigma(L) \circ J^{s+1}\gamma = 0, \quad 1 \le \sigma \le m.$$

2.7. Invariant forms, invariant transformations

A local automorphism α of Y is said to be an *invariant transformation* of a differential form $\eta \in \Omega^r_n W$, if

(12) $$J^r\alpha^*\eta = \eta$$

holds. In this case, η is said to be *invariant* with respect to α.

We say that η is *invariant* with respect to a local 1-parameter group α_t of transformations of Y, if for every $t \in (-\epsilon, \epsilon)$, α_t is an invariant transformation of η. Let the transformations α_t be generated by a vector field ξ. Then the form η is invariant with respect to the family α_t if $\partial_{J^r\xi}\eta = 0$. In that case we also say that ξ is a *generator of invariant transformations* of η.

Consider a variational problem defined by some lagrangian λ for Y. If λ is invariant with respect to the family of α_t, we say that ξ is a generator of invariant transformations of variational problem defined by λ.

2.8. Noether's theorem

The following is an immediate consequence of the first variation formula (8) and (10). Let $\lambda \in \Omega^r_{n,X} Y$ be a lagrangian, and let ξ be a generator of invariant transformations of λ. Let γ be an extremal of the variational problem defined by λ on Y. Then for any Lepage equivalent ρ of λ of order s,

$$dJ^s\gamma^* i_{J^s\xi}\rho = 0$$

(see [9], [15]).

3. Geometry of the frame bundles

3.1. Frame bundles

Let X be an n-dimensional smooth manifold. A *frame* Ξ at a point $x \in X$ is a pair (x, v), where $v = (v_1, v_2, \dots, v_n)$ is a basis of the tangent space T_xX. The set of all frames at the points of X is denoted by FX and it is a fibered manifold with base X and bundle projection $\mu : FX \longrightarrow X$, $\dim FX = n + n^2$. FX has the structure of a right principal $Gl_n(\mathbb{R})$-bundle over X. r-jet prolongation of FX is denoted by J^rFX.

3.2. Charts

Any chart on X defines a chart on FX as follows. If (U,φ), $\varphi=(x^i)$, $i=1,2,\dots,n$, is a chart on X, then the *associated chart* on FX is the pair (V,ψ), $\psi=(x^i,x^i_j)$, $i,j=1,2,\dots,n$, where $V=\mu^{-1}(U)$, and the coordinate functions $x^i,x^i_j : V\longrightarrow\mathbb{R}$ are defined by

$$x^i(\Xi)=x^i(\mu(\Xi)),\qquad v_k=x^i_k(\Xi)\left(\frac{\partial}{\partial x^i}\right)_x.$$

Obviously, $\det(x^i_k)\neq 0$ at every point $\Xi\in V$. We define the inverse matrix y^k_j by setting

$$y^k_j(\Xi)x^i_k(\Xi)=\delta^i_j.$$

If $\Upsilon\in J^rFX$, $\Upsilon=J^r_x\gamma$, where $X\ni x\longrightarrow\gamma(x)=\Xi\in FX$ is a smooth section of FX, then the chart (V^r,ψ^r), $\psi^r=(x^i,x^i_j,x^i_{j,k_1},x^i_{j,k_1k_2},\dots,x^i_{j,k_1k_2\dots k_r})$, on J^rFX, associated with the fibered chart (V,ψ), $\psi=(x^i,x^i_j)$, is defined by

$$x^i(\Upsilon)=x^i(\mu(\Xi)),\qquad x^i_j(\Upsilon)=x^i_j(\Xi),$$
$$x^i_{j,k_1k_2\dots k_m}(\Upsilon)=D_{k_1}D_{k_2}\dots D_{k_m}(x^i_j\circ\gamma\circ\varphi^{-1})(\varphi(x)),\qquad 1\leq m\leq r.$$

3.3. The action of $Gl_n(\mathbb{R})$ on FX and J^rFX

If Ξ is a frame at $x\in X$ and $A\in Gl_n(\mathbb{R})$, then in the canonical coordinates a^i_j on $Gl_n(\mathbb{R})$, the right action $(\Xi,A)\longrightarrow R_\Xi(A)=R_A(\Xi)=\Xi\cdot A$ of $Gl_n(\mathbb{R})$ on FX is expressed, in an obvious sense, by the equations

$$\bar{x}^i=x^i\circ R_A=x^i,\qquad \bar{x}^i_j=x^i_j\circ R_A=x^i_ka^k_j. \tag{13}$$

This action of $Gl_n(\mathbb{R})$ on FX can canonically be prolonged to J^rFX. If $A\in Gl_n(\mathbb{R})$, $\Upsilon\in J^rFX$, $\Upsilon=J^r_x\gamma$, then $\Upsilon\cdot A$ is defined as the r-jet at x of the section $x\longrightarrow R_A(\gamma(x))=\gamma(x)\cdot A$, i.e., $\Upsilon\cdot A=J^r_x(R_A\circ\gamma)$. In the associated chart (V^r,ψ^r), $\psi^r=(x^i,x^i_j,x^i_{j,k_1},x^i_{j,k_1k_2},\dots,\ x^i_{j,k_1k_2\dots k_r})$, this action is expressed by the formulas

$$\bar{x}^i=x^i,\qquad \bar{x}^i_{j,k_1k_2\dots k_m}=x^i_{l,k_1k_2\dots k_m}a^l_j,\qquad 0\leq m\leq r. \tag{14}$$

3.4. $Gl_n(\mathbb{R})$-orbits

Eliminating the group variables a^i_j from the equations $\bar{x}^i_j=x^i_la^l_j$ (13), we obtain

$$\bar{y}^j_m\bar{x}^i_{j,k_1}=y^l_mx^i_{l,k_1},\qquad \bar{y}^j_m\bar{x}^i_{j,k_1k_2}=y^l_mx^i_{l,k_1k_2},\qquad\dots,$$
$$\bar{y}^j_m\bar{x}^i_{j,k_1k_2\dots k_r}=y^l_mx^i_{l,k_1k_2\dots k_r}.$$

Denoting

$$\begin{aligned}&\Gamma^i_{k_1p}=-y^l_px^i_{l,k_1},\qquad \Gamma^i_{k_1k_2p}=-y^l_px^i_{l,k_1k_2},\qquad\dots,\\ &\Gamma^i_{k_1k_2\dots k_rp}=-y^l_px^i_{l,k_1k_2\dots k_r},\end{aligned} \tag{15}$$

we see that these functions are invariants of the group action (14). We get the following equations of the $Gl_n(\mathbb{R})$-orbits in J^rFX

$$\Gamma^i_{k_1p}=c^i_{k_1p},\qquad \Gamma^i_{k_1k_2p}=c^i_{k_1k_2p},\qquad\dots,\qquad \Gamma^i_{k_1k_2\dots k_rp}=c^i_{k_1k_2\dots k_rp},$$

where $c^i_{k_1p}, c^i_{k_1k_2p}, \dots, c^i_{k_1k_2\dots k_rp} \in \mathbb{R}$. The functions $\Gamma^i_{k_1k_2\dots k_jp}$ are *symmetric* in $k_1, k_2, \dots, k_j$.

Now we consider $Gl_n(\mathbb{R})$-invariant functions on J^rFX.

Lemma 1. *Every $Gl_n(\mathbb{R})$-invariant function on J^rFX depends on x^i, $\Gamma^i_{k_1p}$, $\Gamma^i_{k_1k_2p}$, $\dots$, $\Gamma^i_{k_1k_2\dots k_rp}$ only.*

We introduce a chart (V^r, Ψ^r), on J^rFX, adapted to the action (14) of $Gl_n(\mathbb{R})$ on J^rFX. If (V, ψ), $\psi = (x^i, x^i_j)$, is any fibered chart on FX, then $\Psi^r = (x^i, x^i_j, \Gamma^i_{k_1p}, \Gamma^i_{k_1k_2p}, \dots, \Gamma^i_{k_1k_2\dots k_rp})$, where the functions $\Gamma^i_{k_1k_2\dots k_mp}$ are defined by (15), for $1 \le m \le r$. The inverse coordinate transformation is given by the formulas

$$x^i = x^i, \quad x^i_j = x^i_j, \quad x^i_{j,k_1} = -x^l_j\Gamma^i_{k_1l},$$
$$x^i_{j,k_1k_2} = -x^l_j\Gamma^i_{k_1k_2l}, \quad \dots, \quad x^i_{j,k_1k_2\dots k_r} = -x^l_j\Gamma^i_{k_1k_2\dots k_rl}.$$

3.5. Fundamental vector fields

It is easy to determine the fundamental vector fields of the action (13). Let $gl_n(\mathbb{R})$ be the Lie algebra of $Gl_n(\mathbb{R})$. The fundamental vector field ξ on FX, associated with an element $\xi_0 \in gl_n(\mathbb{R})$, is defined by the formula $\xi(\Xi) = T_eR_\Xi \cdot \xi_0$, where e is the identity element of $Gl_n(\mathbb{R})$ (see [10]). Expressing ξ_0 in the form

$$\xi_0 = \xi^i_j \left(\frac{\partial}{\partial a^i_j}\right)_e \tag{16}$$

we get

Lemma 2. *The fundamental vector field ξ on FX, associated with an element $\xi_0 \in gl_n(\mathbb{R})$ (16), has in the associated chart on FX an expression*

$$\xi = \xi^i_s x^r_i \frac{\partial}{\partial x^r_s}. \tag{17}$$

By definition, the r-jet prolongation of the right action of $Gl_n(\mathbb{R})$ on FX (see (14)) generates the fundamental vector fields on J^rFX. We can easily prove the following assertions.

Lemma 3. *The fundamental vector field $\tilde{\xi}$ on J^rFX, associated with an element $\xi_0 \in gl_n(\mathbb{R})$ (16), is expressed in the associated chart (V^r, ψ^r), $\psi^r = (x^i, x^i_j, x^i_{j,k_1}, x^i_{j,k_1k_2}, \dots, x^i_{j,k_1k_2\dots k_r})$, on J^rFX, by the formula*

$$\begin{aligned}\tilde{\xi} = \xi^i_s \bigg(& x^t_i \frac{\partial}{\partial x^t_s} + x^t_{i,k_1}\frac{\partial}{\partial x^t_{s,k_1}} + x^t_{i,k_1k_2}\frac{\partial}{\partial x^t_{s,k_1k_2}} + \dots \\ & + x^t_{i,k_1k_2\dots k_r}\frac{\partial}{\partial x^t_{s,k_1k_2\dots k_r}}\bigg).\end{aligned} \tag{18}$$

Lemma 4. *The r-jet prolongation $J^r\xi$ of a fundamental vector field ξ on FX (17) associated with $\xi_0 \in gl_n(\mathbb{R})$, coincides with the fundamental vector field $\tilde{\xi}$ on J^rFX (18) associated with ξ_0.*

Lemma 5. *The r-jet prolongation $J^r\xi$ of a fundamental vector field ξ on FX, associated with $\xi_0 \in gl_n(\mathbb{R})$ given by (16), has in the coordinate chart (V^r, Ψ^r), $\Psi^r =$*

$(x^i, x^i_j, \Gamma^i_{k_1p}, \Gamma^i_{k_1k_2p}, \dots, \Gamma^i_{k_1k_2\dots k_rp})$, *adapted to the action of* $Gl_n(\mathbb{R})$ *on* J^rFX, *an expression*

$$J^r\xi = \xi^i_s\left(x^t_i\frac{\partial}{\partial x^t_s} + \sum_{j=1}^{r}\sum_{k_1\le k_2\le\dots\le k_j} y^s_l x^p_i \Gamma^t_{k_1k_2\dots k_jp}\frac{\partial}{\partial \Gamma^t_{k_1k_2\dots k_jl}}\right). \tag{19}$$

Remark 1. The construction of a fundamental vector field $\tilde{\xi}$ on J^rFX gives us that ξ is a generator of invariant transformations of differential forms which are invariant with respect to the right action of $Gl_n(\mathbb{R})$ on J^rFX (see Section 2.7.).

3.6. Invariant differential forms

We now consider invariance of differential forms in the sense of Section 2.7. with respect to the transformations induced by the action (14) of $Gl_n(\mathbb{R})$ on J^rFX. We have the following result.

Theorem 1. *A* k*-form* η *on* J^rFX *is* $Gl_n(\mathbb{R})$*-invariant if and only if in any chart on* FX,

$$\begin{aligned}\eta &= \Delta_0 + y^{q_1}_{r_1}dx^{p_1}_{q_1}\wedge\Delta^{r_1}_{p_1} + y^{q_1}_{r_1}y^{q_2}_{r_2}dx^{p_1}_{q_1}\wedge dx^{p_2}_{q_2}\wedge\Delta^{r_1r_2}_{p_1p_2}\\ &\quad + \dots + y^{q_1}_{r_1}y^{q_2}_{r_2}\dots y^{q_k}_{r_k}dx^{p_1}_{q_1}\wedge dx^{p_2}_{q_2}\dots\wedge dx^{p_k}_{q_k}\wedge\Delta^{r_1r_2\dots r_k}_{p_1p_2\dots p_k},\end{aligned} \tag{20}$$

where Δ_0, $\Delta^{r_1}_{p_1}$, $\Delta^{r_1r_2}_{p_1p_2}$, $\dots$, $\Delta^{r_1r_2\dots r_k}_{p_1p_2\dots p_k}$ *are arbitrary forms defined on the quotient space* $C^rX = J^rFX/Gl_n(\mathbb{R})$, *and* $C^0X = X$.

Proof. Let (U,φ), $\varphi = (x^i)$ be a chart on X. In what follows we work with the associated charts (x^i, x^i_j) on FX, $(x^i, x^i_j, x^i_{j,k_1}, x^i_{j,k_1k_2}, \dots, x^i_{j,k_1k_2\dots k_r})$ on J^rFX, and with the chart $(x^i, x^i_j, \Gamma^i_{k_1j}, \Gamma^i_{k_1k_2j}, \dots, \Gamma^i_{k_1k_2\dots k_rj})$, adapted to the action of $Gl_n(\mathbb{R})$ on J^rFX. Recall that in the associated chart, the right action of $Gl_n(\mathbb{R})$ on J^rFX is expressed as in (14) and the functions $\Gamma^i_{k_1p}$, $\Gamma^i_{k_1k_2p}$, $\dots$, $\Gamma^i_{k_1k_2\dots k_rp}$ (15), are invariants of this action.

Consider an *invariant* k-form

$$\begin{aligned}\eta &= \eta_0 + dx^{p_1}_{q_1}\wedge\eta^{q_1}_{p_1} + dx^{p_1}_{q_1}\wedge dx^{p_2}_{q_2}\wedge\eta^{q_1q_2}_{p_1p_2}\\ &\quad + \dots + dx^{p_1}_{q_1}\wedge dx^{p_2}_{q_2}\wedge\dots\wedge dx^{p_k}_{q_k}\wedge\eta^{q_1q_2\dots q_k}_{p_1p_2\dots p_k},\end{aligned}$$

on J^rFX, where η_0, $\eta^{q_1}_{p_1}$, $\eta^{q_1q_2}_{p_1p_2}$, $\dots$, $\eta^{q_1q_2\dots q_k}_{p_1p_2\dots p_k}$ are forms, which do not contain any exterior factor dx^p_q, and the pull-back $(J^rR_A)^*\eta$. Applying the invariance condition $(J^rR_A)^*\eta = \eta$ (12), and the formula $y^s_i\bar{x}^i_q = a^s_q$ we obtain, for every m, $1\le m\le k$,

$$\begin{aligned}&\eta_0 = \bar{\eta}_0 = \Delta_0,\\ &x^{r_1}_{s_1}x^{r_2}_{s_2}\dots x^{r_m}_{s_m}\eta^{s_1s_2\dots s_m}_{p_1p_2\dots p_m} = \bar{x}^{r_1}_{q_1}\bar{x}^{r_2}_{q_2}\dots\bar{x}^{r_m}_{q_m}\bar{\eta}^{q_1q_2\dots q_m}_{p_1p_2\dots p_m} = \Delta^{r_1r_2\dots r_m}_{p_1p_2\dots p_m},\end{aligned}$$

where Δ_0 and $\Delta^{r_1r_2\dots r_m}_{p_1p_2\dots p_m}$ are differential forms which do not contain any exterior factor dx^p_q, and whose components are invariant functions. Then by Lemma 1, these functions depend on x^i, $\Gamma^i_{k_1p}$, $\Gamma^i_{k_1k_2p}$, $\dots$, $\Gamma^i_{k_1k_2\dots k_rp}$ only. In other words, Δ_0, $\Delta^{r_1r_2\dots r_m}_{p_1p_2\dots p_m}$ are differential forms defined on the quotient space $J^rFX/Gl_n(\mathbb{R})$. Indeed, $\eta^{t_1t_2\dots t_m}_{p_1p_2\dots p_m} = y^{t_1}_{r_1}y^{t_2}_{r_2}\dots y^{t_m}_{r_m}\Delta^{r_1r_2\dots r_m}_{p_1p_2\dots p_m}$, and (20) follows.

4. Invariant variational principles for frame bundles

In this section we give a criterion for a lagrangian on J^rFX to be $Gl_n(\mathbb{R})$-invariant. We determine Lepage equivalents, the E-L forms and the Noether's currents corresponding with invariant lagrangians on J^1FX and J^2FX.

4.1. Invariant lagrangians

Theorem 2. *A lagrangian λ on J^rFX is $Gl_n(\mathbb{R})$-invariant if and only if in any chart (V^r, Ψ^r), $\Psi^r = (x^i, x^i_j, \Gamma^i_{k_1j}, \Gamma^i_{k_1k_2j}, \dots, \Gamma^i_{k_1k_2\dots k_rj})$, adapted to the action of $Gl_n(\mathbb{R})$, on J^rFX, it is expressed as $\lambda = L\omega_0$, where L depends on $x^i, \Gamma^i_{k_1j}, \Gamma^i_{k_1k_2j}, \dots, \Gamma^i_{k_1k_2\dots k_rj}$ only.*

Proof. According to Theorem 1, $\lambda = \Delta_0$, where Δ_0 is arbitrary form on $C^rX = J^rFX/Gl_n(\mathbb{R})$, so L does not depend on coordinates x^i_j.

4.2. Lepage equivalents of invariant lagrangians

For applications, it is convenient to work with Lepage equivalents of a $Gl_n(\mathbb{R})$-invariant lagrangian in the chart adapted to the action of $Gl_n(\mathbb{R})$ on J^rFX. We give the chart expressions for the Lepage equivalents of lagrangians of orders 1 and 2, discussed in Section 2

The following assertion is used in the proofs.

Lemma 6. *Let f be an invariant function on J^rFX. Then p-th formal derivative d_pf of f is invariant function on $J^{r+1}FX$ and in chart (V^{r+1}, Ψ^{r+1}), $V \subset W$, on $J^{r+1}FX$, it can be expressed by*

$$d_pf = \frac{\partial f}{\partial x^p} + \sum_{s=1}^{r} \frac{\partial f}{\partial \Gamma^i_{k_1k_2\dots k_sj}} \Gamma^i_{k_1k_2\dots k_spj}.$$

Let us define

$$\omega^i_{j,k_1k_2\dots k_s} = dx^i_{j,k_1k_2\dots k_s} - x^i_{j,k_1k_2\dots k_sm}dx^m, \quad 0 \le s \le r.$$

The following lemmas can be proved, with the help of Section 2 (see (1), (2), (3), (4)), by a straightforward calculation.

Lemma 7. *Let $\lambda \in \Omega^1_{n,X}W$ be an invariant lagrangian. Then the Poincaré-Cartan form $\Theta_\lambda \in \Omega^1_{n,Y}W$ of λ is expressed in adapted coordinates by*

$$\Theta_\lambda = \left(L - \Gamma^i_{kl}\frac{\partial L}{\partial \Gamma^i_{kl}}\right)\omega_0 - y^j_l \frac{\partial L}{\partial \Gamma^i_{kl}} dx^i_j \wedge \omega_k, \tag{21}$$

where $\lambda = L\omega_0$.

Corollary 1. *The Poincaré-Cartan form Θ_λ of invariant lagrangian λ is invariant differential form on J^1FX.*

Proof. We can write (21) in the form $\Theta_\lambda = \Delta_0 + y^j_l dx^i_j \wedge \Delta^l_i$, where the differential forms

$$\Delta_0 = \left(L - \Gamma^i_{kl}\frac{\partial L}{\partial \Gamma^i_{kl}}\right)\omega_0, \quad \Delta^l_i = -\frac{\partial L}{\partial \Gamma^i_{kl}}\omega_k$$

are defined on $C^1X = J^1FX/Gl_n(\mathbb{R})$. By Theorem 1 it means that Θ_λ is invariant differential form on J^1FX.

Lemma 8. *If $\lambda \in \Omega^1_{n,X}W$ is invariant lagrangian, then the fundamental Lepage equivalent Φ_λ of this lagrangian $\lambda = L\omega_0$ is in adapted coordinates given by*

$$\begin{aligned}\Phi_\lambda = {} & \sum_{k=0}^{n} \frac{1}{k!\,(n-k)!} \frac{\partial^k L}{\partial \Gamma^{i_1}_{l_1m_1} \partial \Gamma^{i_2}_{l_2m_2} \dots \partial \Gamma^{i_k}_{l_km_k}} (-1)^k \epsilon_{l_1l_2\dots l_ki_{k+1}i_{k+2}\dots i_n} \\ & \cdot \sum_{p=0}^{k} \frac{1}{p!\,(k-p)!} y^{j_1}_{m_1} y^{j_2}_{m_2} \dots y^{j_p}_{m_p} \Gamma^{i_{p+1}}_{s_{p+1}m_{p+1}} \Gamma^{i_{p+2}}_{s_{p+2}m_{p+2}} \dots \Gamma^{i_k}_{s_km_k} \\ & \cdot dx^{i_1}_{j_1} \wedge dx^{i_2}_{j_2} \wedge \dots \wedge dx^{i_p}_{j_p} \wedge dx^{s_{p+1}} \wedge dx^{s_{p+2}} \wedge \dots \wedge dx^{s_k} \\ & \wedge dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \dots \wedge dx^{i_n}.\end{aligned} \tag{22}$$

Corollary 2. *The fundamental Lepage equivalent Φ_λ of invariant lagrangian λ is invariant differential form on J^1FX.*

Proof. It is evident that for every k, the corresponding part of Φ_λ (22) has an expression required by Theorem 1 for being an invariant differential form on J^1FX.

Lemma 9. *Let $\lambda \in \Omega^2_{n,X}W$ be an invariant lagrangian. Then the Lepage equivalent $\Theta_\lambda \in \Omega^3_nW$ of λ is expressed in adapted chart by*

$$\begin{aligned}\Theta_\lambda = {} & \left(L - \Gamma^i_{kl}\frac{\partial L}{\partial \Gamma^i_{kl}} + \Gamma^i_{kl}\left(\Gamma^l_{pq}\frac{\partial L}{\partial \Gamma^i_{pkq}} + d_p\frac{\partial L}{\partial \Gamma^i_{pkl}}\right) - \Gamma^i_{lkt}\frac{\partial L}{\partial \Gamma^i_{lkt}}\right)\omega_0 \\ & + \frac{\partial L}{\partial \Gamma^i_{lkt}} d\Gamma^i_{lt} \wedge \omega_k \\ & + y^j_l\left(\Gamma^t_{pi}\frac{\partial L}{\partial \Gamma^t_{pkl}} + \Gamma^l_{pq}\frac{\partial L}{\partial \Gamma^i_{pkq}} + d_p\frac{\partial L}{\partial \Gamma^i_{pkl}} - \frac{\partial L}{\partial \Gamma^i_{kl}}\right) dx^i_j \wedge \omega_k,\end{aligned} \tag{23}$$

where $\lambda = L\omega_0$.

Corollary 3. *The Lepage equivalent Θ_λ of invariant lagrangian λ on J^2FX is invariant differential form on J^3FX.*

Proof. The form Θ_λ can be rewritten in the form $\Theta_\lambda = \Delta_0 + y^j_l dx^i_j \wedge \Delta^l_i$, where forms Δ_0, and Δ^l_i depend on $x^i, \Gamma^i_{k_1j}, \Gamma^i_{k_1k_2j}, \Gamma^i_{k_1k_2k_3j}$. It means that, by Theorem 1, Θ_λ is invariant differential form.

4.3. Euler-Lagrange forms of invariant lagrangians

We give explicit formulas for the Euler-Lagrange forms associated with invariant lagrangians of the first and the second order.

Lemma 10. *Let λ be an invariant lagrangian on J^1FX. If in a chart (U, φ), $\varphi = (x^i)$, on X, $\lambda = L\omega_0$, then E_λ associated with λ has in the adapted chart on J^2FX expression*

$$E_\lambda = y^j_l\left(\Gamma^l_{pq}\frac{\partial L}{\partial \Gamma^i_{pq}} + \frac{\partial^2 L}{\partial \Gamma^i_{pl}\partial x^p} + \frac{\partial^2 L}{\partial \Gamma^i_{pl}\Gamma^k_{mc}}\Gamma^k_{mpc}\right) dx^i_j \wedge \omega_0. \tag{24}$$

Corollary 4. *The Euler-Lagrange form E_λ associated with an invariant lagrangian $\lambda \in \Omega^1_{n,X}W$ is invariant differential $(n+1)$-form on J^2FX.*

Proof. We can write $E_\lambda = y^j_l dx^i_j \wedge \Delta^l_i$, where Δ^l_i are defined on $C^2X = J^2FX/Gl_n(\mathbb{R})$, so we can use Theorem 1 again.

Lemma 11. *Let λ be an invariant lagrangian on J^2FX. If λ is expressed by $\lambda = L\omega_0$, then in the adapted chart (V^4, Ψ^4) on J^4FX,*

$$E_\lambda = y^j_l g^l_i dx^i_j \wedge \omega_0,$$

where

$$g^l_i = \Gamma^l_{pq}\frac{\partial L}{\partial \Gamma^i_{pq}} + d_p \frac{\partial L}{\partial \Gamma^i_{pl}} - \Gamma^l_{pr}\Gamma^r_{qt}\frac{\partial L}{\partial \Gamma^i_{pqt}} - \Gamma^l_{pqt}\frac{\partial L}{\partial \Gamma^i_{pqt}} - 2\Gamma^l_{qt} d_p \frac{\partial L}{\partial \Gamma^i_{pqt}} - d_p d_q \frac{\partial L}{\partial \Gamma^i_{pql}}.$$

Corollary 5. *The Euler-Lagrange form E_λ associated with invariant lagrangian $\lambda \in \Omega^2_{n,X}W$ is invariant differential $(n+1)$-form on J^4FX.*

Proof. Taking into account that forms $\Delta^l_i = g^l_i\omega_0$ are defined on C^4X, by Theorem 1, E_λ is invariant differential form on J^4FX.

4.4. Noether's currents

Let λ be a $Gl_n(\mathbb{R})$-invariant lagrangian on J^rFX and let ρ be any Lepage equivalent of λ, defined on J^sFX. Suppose that we have a fundamental vector field ξ on FX; we know that ξ is the generator of invariant transformations of λ. If γ is an extremal then from the infinitesimal first variation formula (8) and from (10),

$$dJ^s\gamma^* i_{J^s\xi}\rho = 0. \tag{25}$$

The $(n-1)$-form $\Psi_{\xi,\rho} = i_{J^s\xi}\rho$ is called the *Noether's current* associated with ξ and the Lepage equivalent ρ.

In this subsection we derive explicit chart expressions for the Noether's currents for $Gl_n(\mathbb{R})$-invariant lagrangians of order 1 and 2.

Theorem 3. *Let λ be an invariant lagrangian on J^1FX and let Θ_λ be its Poincaré-Cartan form. Then in any adapted chart, the Noether's current $\Psi_{\xi,\Theta_\lambda}$ associated with a fundamental vector field ξ and Θ_λ is given by*

$$\Psi_{\xi,\Theta_\lambda} = -\xi^m_j y^j_l x^i_m \frac{\partial L}{\partial \Gamma^i_{kl}}\omega_k. \tag{26}$$

Proof. To prove Theorem 3, we compute $i_{J^1\xi}\Theta_\lambda$ in adapted chart for $J^1\xi$ given by (19), and Θ_λ given by (21).

Theorem 4. *Let λ be a $Gl_n(\mathbb{R})$-invariant lagrangian on J^2FX and Θ_λ be its Lepage equivalent given by (23). Then in any adapted chart, Noether's current $\Psi_{\xi,\Theta_\lambda}$ associated with a fundamental vector field ξ and Θ_λ is given by*

$$\Psi_{\xi,\Theta_\lambda} = \xi^m_j y^j_l x^i_m \left(2\Gamma^q_{pi}\frac{\partial L}{\partial \Gamma^q_{pkl}} + \Gamma^l_{pq}\frac{\partial L}{\partial \Gamma^i_{pkq}} + d_p\frac{\partial L}{\partial \Gamma^i_{pkl}} - \frac{\partial L}{\partial \Gamma^i_{kl}}\right)\omega_k. \tag{27}$$

Proof. Consider the Lepage equivalent Θ_λ (23). Denote

$$f = L - \Gamma^i_{kl}\frac{\partial L}{\partial \Gamma^i_{kl}} + \Gamma^i_{kl}\left(\Gamma^l_{pq}\frac{\partial L}{\partial \Gamma^i_{pkq}} + d_p\frac{\partial L}{\partial \Gamma^i_{pkl}}\right) - \Gamma^i_{lkt}\frac{\partial L}{\partial \Gamma^i_{lkt}},$$

and

$$F^{kl}_i = \Gamma^t_{pi}\frac{\partial L}{\partial \Gamma^t_{pkl}} + \Gamma^l_{pq}\frac{\partial L}{\partial \Gamma^i_{pkq}} + d_p\frac{\partial L}{\partial \Gamma^i_{pkl}} - \frac{\partial L}{\partial \Gamma^i_{kl}}.$$

Then

$$\Theta_\lambda = f\omega_0 + \frac{\partial L}{\partial \Gamma^i_{lkt}} d\Gamma^i_{lt}\wedge\omega_k + y^j_l F^{kl}_i dx^i_j \wedge \omega_k.$$

We have to compute its contraction $i_{J^2\xi}\Theta_\lambda$ by fundamental vector field $J^2\xi$ given by (19). Using these expressions we get (27).

To illustrate the meaning of Theorem 3 we show how the Noether's currents can be used for a reduction of the E-L equations.

Theorem 5. *Let λ be an invariant lagrangian on J^1FX, expressed, in any chart on X, as $\lambda = L\omega_0$.*

(a) *Let* $\dim X = 1$. *The Euler-Lagrange equation $E(L)\circ J^2\gamma = 0$ is equivalent with the equation $F(L)\circ J^1\gamma = 0$ of the first order, where, in the corresponding adapted chart,*

$$F(L) = \Gamma\frac{\partial}{\partial\Gamma}\left(\frac{\partial L}{\partial \Gamma}\Gamma - 2L\right). \tag{28}$$

(b) *Let* $\dim X \geq 2$. *The system of n^2 Euler-Lagrange equations $E^j_i(L)\circ J^2\gamma = 0$ of the variational problem defined by λ, can be reduced to the system of n^2 partial differential equations $F^l_i(L)\circ J^1\gamma = 0$ of the first order, where, in the corresponding adapted chart,*

$$F^l_i(L) = \frac{\partial^2 L}{\partial \Gamma^i_{kl}\Gamma^p_{qc}}\Gamma^p_{qs}\Gamma^s_{kc} + \frac{\partial L}{\partial \Gamma^p_{kl}}\Gamma^p_{ki}. \tag{29}$$

Proof. (a) Let $\dim X = 1$. According to Theorem 2, an invariant lagrangian on J^1FX has in adapted coordinates (t, x, Γ) an expression $\lambda = Ldt$, where $L = L(t,\Gamma)$. By (17), we have only one linearly independent fundamental vector field, we can choose $\xi = x(\partial/\partial x)$. Let us denote the corresponding Noether's currents $\Psi_{\xi,\Theta_\lambda} = \Psi$. By (26) in Theorem 3,

$$\Psi = -\frac{\partial L}{\partial \Gamma}.$$

Then

$$d\Psi = d\left(-\frac{\partial L}{\partial\Gamma}\right) = -\frac{\partial^2 L}{\partial\Gamma\partial t}dt - \frac{\partial^2 L}{\partial \Gamma^2}d\Gamma$$

and along each extremal γ,

$$dJ^1\gamma^*\Psi = J^1\gamma^* d\Psi = -\left(\frac{\partial^2 L}{\partial\Gamma\partial t} + \frac{\partial^2 L}{\partial\Gamma^2}\Gamma^2 + \frac{\partial^2 L}{\partial \Gamma^2}\dot\Gamma\right)dt = -H(L)dt,$$

where $\dot\Gamma = -y\ddot{x}$. By (25) we have $H(L)\circ J^2\gamma = 0$ for each extremal γ.

If we denote $E(L) = yB(L)$, (11) implies $B(L)\circ J^2\gamma = 0$ for each extremal γ, where, by (24),

$$B(L) = \frac{\partial L}{\partial \Gamma}\Gamma + \frac{\partial^2 L}{\partial \Gamma \partial t} + \frac{\partial^2 L}{\partial \Gamma^2}\dot{\Gamma}.$$

Setting $F(L) = H(L) - B(L)$ we get assertion (a) of Theorem 5.

(b) Let $\dim X \geq 2$. According to Theorem 2, an invariant lagrangian on J^1FX has in adapted coordinates expression $\lambda = L\omega_0$, where $L = L(x^i, \Gamma^i_{kj})$. By (17), we have n^2 linearly independent fundamental vector fields $\xi^j_m = x^t_m(\partial/\partial x^t_j)$. Let us denote the corresponding Noether's currents $\Psi_{\xi^j_m,\Theta_\lambda} = \Psi^j_m$, $1 \leq j, m \leq n$. By (26) in Theorem 3,

$$\Psi^j_m = -y^j_l x^i_m \frac{\partial L}{\partial \Gamma^i_{kl}} \omega_k.$$

Then

$$\begin{aligned} d\Psi^j_m &= d\left(-y^j_l x^i_m \frac{\partial L}{\partial \Gamma^i_{kl}} \omega_k\right) \\ &= y^j_p y^q_l x^i_m \frac{\partial L}{\partial \Gamma^i_{kl}} dx^p_q \wedge \omega_k - y^j_l \frac{\partial L}{\partial \Gamma^i_{kl}} dx^i_m \wedge \omega_k \\ &\quad - y^j_l x^i_m \frac{\partial^2 L}{\partial \Gamma^i_{kl} \partial x^k} \omega_0 - y^j_l x^i_m \frac{\partial^2 L}{\partial \Gamma^i_{kl} \partial \Gamma^p_{qc}} d\Gamma^p_{qc} \wedge \omega_k, \end{aligned}$$

and along each extremal γ,

$$\begin{aligned} dJ^1\gamma^* \Psi^j_m &= J^1\gamma^* d\Psi^j_m \\ &= -y^j_l \left(x^i_m \frac{\partial L}{\partial \Gamma^i_{kp}} \Gamma^l_{kp} - \frac{\partial L}{\partial \Gamma^i_{kl}} x^i_{m,k} + x^i_m \frac{\partial^2 L}{\partial \Gamma^i_{kl} \partial x^k} \right. \\ &\quad \left. + x^i_m \frac{\partial^2 L}{\partial \Gamma^i_{kl} \partial \Gamma^p_{qc}} \Gamma^p_{qs} \Gamma^s_{kc} + x^i_m \frac{\partial^2 L}{\partial \Gamma^i_{kl} \partial \Gamma^p_{qc}} \Gamma^p_{qkc} \right) \omega_0 \\ &= -y^j_l G^l_m(L) \omega_0. \end{aligned}$$

Denoting $H^l_i(L) = y^m_i G^l_m(L)$, we obtain

$$\begin{aligned} H^l_i(L) &= \frac{\partial L}{\partial \Gamma^i_{kp}} \Gamma^l_{kp} + \frac{\partial L}{\partial \Gamma^p_{kl}} \Gamma^p_{ki} + \frac{\partial^2 L}{\partial \Gamma^i_{kl} \partial x^k} \\ &\quad + \frac{\partial^2 L}{\partial \Gamma^i_{kl} \partial \Gamma^p_{qc}} \Gamma^p_{qs} \Gamma^s_{kc} + \frac{\partial^2 L}{\partial \Gamma^i_{kl} \partial \Gamma^p_{qc}} \Gamma^p_{qkc}. \end{aligned}$$

By (25) we have $H^l_i(L) \circ J^2\gamma = 0$ for each extremal γ.

If we denote $E^j_i(L) = y^j_l B^l_i(L)$, (11) implies $B^l_i(L) \circ J^2\gamma = 0$ for each extremal γ, where, by (24),

$$B^l_i(L) = \frac{\partial L}{\partial \Gamma^i_{kp}} \Gamma^l_{kp} + \frac{\partial^2 L}{\partial \Gamma^i_{kl} \partial x^k} + \frac{\partial^2 L}{\partial \Gamma^i_{kl} \Gamma^p_{qc}} \Gamma^p_{qkc}.$$

Setting $F^l_i(L) = H^l_i(L) - B^l_i(L)$ we get assertion (b) of Theorem 5.

Remark 2. The procedure similar to one shown in the Proof of Theorem 5 can be easily applied to second order variational problems. Due to difficult expressions of the Noether's currents and the E-L form of the second order lagrangian we do not give the corresponding formulas.

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Ján Brajerčík
Department of Mathematics FHPV PU
17. nov. 1, 081 16 Prešov, Slovakia
e-mail: brajerci@unipo.sk

Demeter Krupka
Department of Algebra and Geometry
Palacky University, Tomkova 40
77900 Olomouc, Czech Republic
and
Department of Mathematics
La Trobe University, Melbourne
Victoria 3086, Australia
e-mail: krupka@inf.upol.cz