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Variational sequences and Lepage forms

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Abstract. The concept of a Lepage form introduced by Krupka is generalized to differential forms of arbitrary degree in higher order mechanics. It is shown, that the Lepage forms can be used to find a natural representation of classes of forms in variational sequences as globally well-defined differential forms. The resulting expressions for the classes agree with Krbek and Musilová. The construction of Lepage forms is based on the interior Euler operator as introduced by Anderson within the variational bicomplex theory; we give a straightforward finite order definition of this operator, and prove its invariance. The structure of Lepage equivalents of the Euler-Lagrange forms is discussed in detail.

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1. Introduction

The aim of this paper is to complete some recent results on possible generalizations of the concept of a Lepage form to forms of arbitrary degree in higher order mechanics [13], i.e. in fibered manifolds over one dimensional bases.

The Lepage forms, as well as Lepage equivalents of lagrangians for multidimensional variational problems in fibered manifolds, were introduced by Krupka [6], [7]. These forms can be used for an invariant description of the first variational formula, and some global concepts such as the Euler-Lagrange form and the Noether current. The Lepage forms unify several interesting objects in the calculus of variations, among them the Poincare-Cartan form, the Caratheodory form and the so called fundamental form, and give us a generalization of these forms to higher order variational theory. They also represent a fundamental notion for possible generalizations of the Hamilton theory to higher order problems in fibered spaces.

In the context of the variational sequence theory, the exterior derivative of a Lepage form immediately gives the corresponding class, which turns up to be the Euler-Lagrange form.

Our main motivation for possible generalizations is to achieve an analogous property for forms of degree higher then 1. In particular, we show that the exterior derivative of a

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Lepage 2-form will give us, as its class, the well known (invariant) Helmholtz form (see also Krupka [5], Šeděnková [13]).

Recently, Krupková presented an extension of the notion of a Lepage form to 2-forms (in mechanics) and to $(n+1)$-forms (in field theory), and applied these forms to the inverse problem in higher order mechanics, and to the order reducibility problem. She defined the Lepage forms as the *closed* counterpart of the Euler-Lagrange form (see [10], [11]).

Our definition agrees with the structure of variational sequences (Krbek, Musilová [4], Krupka [8], [9], Šeděnková [14], [15], Vitolo [16]) and is more general. We use a (finite order) modification of the interior Euler operator as presented by Anderson [1] for the variational bicomplex (for this operator see also Bauderon [2], Dedecker and Tulczyjew [3], Kuperschmidt [12]). We give independently a simple proof of basic properties of this operator.

2. Preliminaries

Let $\pi : Y \longrightarrow X$ be a fibered manifold with fibered coordinate systems (V,ψ), $\psi = (t, q^\sigma)$, on Y and (U,φ), $\varphi = (t)$ on X, $\dim X = 1$, $\dim Y = m+1$. Denote by $\pi^r : J^rY \longrightarrow X$ or just J^rY the r-jet prolongation of the fibered manifold $\pi : Y \longrightarrow X$, the coordinate system is (V^r, ψ^r), $\psi^r = (t, q^\sigma, q_1^\sigma, \dots, q_r^\sigma)$ on J^rY. For small r we denote $q_0^\sigma = q^\sigma$, $q_1^\sigma = \dot{q}^\sigma$, $q_2^\sigma = \ddot{q}^\sigma$. The canonical jet projections are $\pi^{r,s} : J^rY \longrightarrow J^sY$, $r > s$ and $\pi^{r,0} : J^rY \longrightarrow Y$.

A differential k-form ρ on J^rY is called *contact*, if it vanishes along the r-jet prolongation $J^r\gamma$ of every section γ of π.

If (V,ψ), $\psi = (t, q^\sigma)$, is a fibered chart on Y, then we often use the *contact basis* $dt, \omega^\sigma, \omega_1^\sigma, \dots, \omega_r^\sigma, dq_{r+1}^\sigma$ on $V^{r+1} = (\pi^{r+1,0})^{-1}V$ given by the forms

(1) $$\omega_j^\sigma = dq_j^\sigma - q_{j+1}^\sigma dt, \qquad 0 \le j \le r.$$

Recall that a form which contains exactly k expressions (1) is called k-contact. Every form ρ on J^rY can be uniquely decomposed, after the lifting to $J^{r+1}Y$, as the sum of the k-contact components $p_k\rho$.

A vector field Ξ on π is called π-vertical if $T\pi \cdot \Xi = 0$. The s-jet prolongation of a π-vertical vector field Ξ on Y, $\Xi = \Xi^\sigma(\partial/\partial q^\sigma)$, is a vector field on J^sY given by

(2) $$J^s\Xi = \sum_{i=0}^{s} \Xi_i^\sigma \frac{\partial}{\partial q_i^\sigma}, \quad \text{where} \quad \Xi_i^\sigma = \frac{d}{dt}\Xi_{i-1}^\sigma.$$

Let Ω_k^r be the direct image of the sheaf of smooth k-forms over J^rY by the jet projection $\pi^{r,0}$, where $k \ge 0$. Denote

(3) $$\Omega_{0,c}^r = \{0\}, \quad \Omega_{k,c}^r = \ker p_{k-1}, \quad \Theta_k^r = \Omega_{k,c}^r + d\Omega_{k-1,c}^r,$$

where $k \ge 1$, and $d\Omega_{k-1,c}^r$ is the image sheaf of $\Omega_{k-1,c}^r$ by d. Then for every open set $W \subset Y$, $\Omega_k^r W$ (resp. $\Omega_{k,c}^r W$) is the Abelian group of k-forms (resp. k-contact k-forms) on $W^r = (\pi^{r,0})^{-1}(W)$, $d\Omega_{k-1,c}^r W$ is the Abelian group of forms which can be locally expressed as differentials of $(k-1)$-contact $(k-1)$-forms on W^r, and $\Theta_k^r W$ is a subgroup of $\Omega_k^r W$. We get a sequence

(4) $$0 \longrightarrow \Theta_1^r \longrightarrow \Theta_2^r \longrightarrow \Theta_3^r \longrightarrow \cdots \longrightarrow \Theta_M^r \longrightarrow 0,$$

in which all arrows denote the exterior differentiation d, and $M = mr + 1$. (4) is a subsequence of the De Rham sequence

$$0 \longrightarrow \mathbb{R}_Y \longrightarrow \Omega_0^r \longrightarrow \Omega_1^r \longrightarrow \Omega_2^r \longrightarrow \cdots \longrightarrow \Omega_{N-1}^r \longrightarrow \Omega_N^r \longrightarrow 0, \tag{5}$$

where $N = \dim J^rY = 1 + m(r+1)$. The quotient sequence

$$\begin{aligned} &0 \longrightarrow \mathbb{R}_Y \longrightarrow \Omega_0^r \longrightarrow \Omega_1^r/\Theta_1^r \longrightarrow \Omega_2^r/\Theta_2^r \longrightarrow \cdots \\ &\quad \cdots \longrightarrow \Omega_M^r/\Theta_M^r \longrightarrow \Omega_{M+1}^r \longrightarrow \cdots \longrightarrow \Omega_{N-1}^r \longrightarrow \Omega_N^r \longrightarrow 0 \end{aligned} \tag{6}$$

is also exact. (6) is called the *r-th order variational sequence*. The class of a differential form $\rho \in \Omega_k^r W$ in the variational sequence (6) is denoted by $[\rho]$.

The quotient mapping $E : \Omega_k^r/\Theta_k^r \longrightarrow \Omega_{k+1}^r/\Theta_{k+1}^r$ is defined by

$$E([\rho]) = [d\rho]. \tag{7}$$

This mapping satisfies the condition $E^2 = 0$. The quotient mapping $E : \Omega_1^r/\Theta_1^r \longrightarrow \Omega_2^r/\Theta_2^r$ is called the *Euler-Lagrange mapping.* The quotient mapping $E : \Omega_2^r/\Theta_2^r \longrightarrow \Omega_3^r/\Theta_3^r$ is called the *Helmholtz-Sonin mapping.*

A *lagrangian* of order r is π^r-horizontal n-form λ. In coordinates, we can write

$$\lambda = L dt, \tag{8}$$

where L is a function on J^rY called *Lagrange function.*

Let ρ be a 1-form on J^rY. A form ρ is called a *Lepage 1-form* if $p_1 d\rho$ is a $\pi^{r+1,0}$-horizontal 2-form. A Lepage form ρ is called a *Lepage equivalent* of a lagrangian λ if $h\rho = \lambda$. It is known that in higher order mechanics, Lepage equivalents are uniquely determined by lagrangians. We denote by θ_λ the Lepage equivalent of a lagrangian lambda. If $r = 1$, θ_λ is the well known *Cartan form*, if $r > 1$, we have the *generalized Poincaré-Cartan form.* If in a fibered chart $\lambda = L\,dt$, then

$$p_1 d\theta_\lambda = E_\sigma(L)\omega^\sigma \wedge dt, \tag{9}$$

where

$$E_\sigma(L) = \sum_{l=0}^{r} (-1)^l \frac{d^l}{dt^l} \frac{\partial L}{\partial q_l^\sigma}. \tag{10}$$

The form (9) is called the *Euler-Lagrange form* and it is denoted by E_λ. The components (10) are called the *Euler-Lagrange expressions.*

3. Interior Euler-Lagrange operator

Our goal in this section is to give a new coordinate-free definition of the interior Euler-Lagrange operator. First we need the following lemma.

Lemma 1. *Let ρ be a 1-contact 2-form on J^rY. There exists a unique 1-contact, ω^σ-generated 2-form ρ_0 on $J^{2r+1}Y$ and a unique contact 1-form ρ' on $J^{2r}Y$ such that*

$$(\pi^{2r+1,r})^*\rho = \rho_0 + p_1 d\rho'. \tag{11}$$

Proof. Let (V^{2r+1}, ψ^{2r+1}) be the associated fibered chart. Let ρ be a 1-contact 2-form on J^rY. Then $(\pi^{2r+1,r})^*\rho$ can be expressed in the form

$$(\pi^{2r+1,r})^*\rho = \sum_{i=0}^{r} P_\sigma^i \omega_i^\sigma \wedge dt. \tag{12}$$

Let us stress that by a direct computation the following property holds

$$p_1 d(P^i_\sigma \omega^\sigma_{i-1}) = -P^i_\sigma \omega^\sigma_i \wedge dt - \frac{d}{dt} P^i_\sigma \omega^\sigma_{i-1} \wedge dt. \tag{13}$$

We compute $(\pi^{2r+1,r})^* \rho$ in such a way that every term which contains $\omega^\sigma_i \wedge dt$, $1 \leq i$, is replaced by

$$\widetilde{P}^i_\sigma \omega^\sigma_i \wedge dt = -\frac{d}{dt} \widetilde{P}^i_\sigma \omega^\sigma_{i-1} \wedge dt - p_1 d(\widetilde{P}^i_\sigma \omega^\sigma_{i-1}). \tag{14}$$

This procedure can be repeated until we eliminate all terms with $\omega^\sigma_i \wedge dt$, $1 \leq i$. Then we obtain

$$(\pi^{2r+1,r})^* \rho = \sum_{i=0}^{r} (-1)^i \frac{d^i}{dt^i} P^i_\sigma \omega^\sigma \wedge dt + p_1 d\left(\sum_{i=1}^{r} \sum_{j=1}^{i} (-1)^j \frac{d^{j-1}}{dt^{j-1}} P^i_\sigma \omega^\sigma_{i-j} \right), \tag{15}$$

and we denote

$$\rho_0 = \sum_{i=0}^{r} (-1)^i \frac{d^i}{dt^i} P^i_\sigma \omega^\sigma \wedge dt, \tag{16}$$

$$\rho' = \sum_{i=1}^{r} \sum_{j=1}^{i} (-1)^j \frac{d^{j-1}}{dt^{j-1}} P^i_\sigma \omega^\sigma_{i-j}. \tag{17}$$

It is evident, that ρ_0 is a 1-contact ω^σ-generated 2-form on $J^{2r+1}Y$ and ρ' is a contact 1-form on $J^{2r}Y$ and (11) is fulfilled.

To complete the proof it is enough to show that this decomposition is unique. Suppose that

$$(\pi^{2r+1,r})^* \rho = \rho_0 + p_1 d\rho' = \bar{\rho}_0 + p_1 d\bar{\rho}'. \tag{18}$$

Denote by

$$\rho_0 = A_\sigma \omega^\sigma \wedge dt,\ \rho' = \sum_{i=0}^{2r-1} B^i_\sigma \omega^\sigma_i,\ \bar{\rho}_0 = \bar{A}_\sigma \omega^\sigma \wedge dt,\ \bar{\rho}' = \sum_{i=0}^{2r-1} \bar{B}^i_\sigma \omega^\sigma_i. \tag{19}$$

Then

$$\begin{aligned} \rho_0 + p_1 d\rho' - \bar{\rho}_0 - p_1 d\bar{\rho}' &= \left(A_\sigma - \bar{A}_\sigma - \frac{d}{dt}(B^0_\sigma - \bar{B}^0_\sigma) \right) \omega^\sigma \wedge dt \\ &- \sum_{i=1}^{2r-1} \left(\frac{d}{dt}(B^i_\sigma - \bar{B}^i_\sigma) + (B^{i-1} i_\sigma - \bar{B}^{i-1}_\sigma) \omega^\sigma_i \wedge dt \right) \\ &- (B^{2r-1}_\sigma - \bar{B}^{2r-1}_\sigma) \omega^\sigma_{2r} \wedge dt \\ &= 0. \end{aligned}$$

We have

$$B^{2r-1}_\sigma - \bar{B}^{2r-1}_\sigma = 0, \quad \dots, \quad B^0_\sigma - \bar{B}^0_\sigma = 0, \quad A_\sigma - \bar{A}_\sigma = 0, \tag{20}$$

i.e.

$$\rho_0 = \bar{\rho}_0, \quad \rho' = \bar{\rho}'. \tag{21}$$

and the decomposition (11) is unique. This completes the proof.

Now we are in a position to introduce two operators I and J, acting on contact forms, in three steps:

(a) If ρ is a 1-contact 2-form on J^rY, we set

$$I(\rho) = \rho_0, \tag{22}$$
$$J(\rho) = \rho'. \tag{23}$$

(b) Let ρ be a 2-contact 3-form on J^rY, let Ξ_1, Ξ_2 be π-vertical vector fields on Y. We define a 2-contact 3-form $I(\rho)$ on $J^{2r+1}Y$ by

$$i_{J^{2r+1}\Xi_2} i_{J^{2r+1}\Xi_1} I(\rho) = \frac{1}{2}\,(i_{J^{2r+1}\Xi_2} I(i_{J^r\Xi_1}\rho) - i_{J^{2r+1}\Xi_1} I(i_{J^r\Xi_2}\rho)), \tag{24}$$

and a 2-contact 2-form $J(\rho)$ on $J^{2r}Y$ by

$$i_{J^{2r}\Xi_2} i_{J^{2r}\Xi_1} J(\rho) = -\frac{1}{2}\,(i_{J^{2r}\Xi_2} J(i_{J^r\Xi_1}\rho) - i_{J^{2r}\Xi_1} J(i_{J^r\Xi_2}\rho)). \tag{25}$$

(c) Let ρ be a k-contact $(k+1)$-form on J^rY, let $\Xi_1, \Xi_2, \dots, \Xi_k$ be π-vertical vector fields on Y. We define a k-contact $(k+1)$-form $I(\rho)$ on $J^{2r+1}Y$ by

$$\begin{aligned}
&i_{J^{2r+1}\Xi_k} \cdots i_{J^{2r+1}\Xi_2} i_{J^{2r+1}\Xi_1} I(\rho)\\
&= \frac{1}{k}(i_{J^{2r+1}\Xi_k} i_{J^{2r+1}\Xi_{k-1}} \cdots i_{J^{2r+1}\Xi_3} i_{J^{2r+1}\Xi_2} I(i_{J^r\Xi_1}\rho)\\
&- i_{J^{2r+1}\Xi_k} i_{J^{2r+1}\Xi_{k-1}} \cdots i_{J^{2r+1}\Xi_3} i_{J^{2r+1}\Xi_1} I(i_{J^r\Xi_2}\rho)\\
&- i_{J^{2r+1}\Xi_k} i_{J^{2r+1}\Xi_{k-1}} \cdots i_{J^{2r+1}\Xi_1} i_{J^{2r+1}\Xi_2} I(i_{J^r\Xi_3}\rho)\\
&- \cdots\\
&- i_{J^{2r+1}\Xi_k} i_{J^{2r+1}\Xi_1} \cdots i_{J^{2r+1}\Xi_1} i_{J^{2r+1}\Xi_2} I(i_{J^r\Xi_{k-1}}\rho)\\
&- i_{J^{2r+1}\Xi_1} i_{J^{2r+1}\Xi_{k-1}} \cdots i_{J^{2r+1}\Xi_3} i_{J^{2r+1}\Xi_2} I(i_{J^r\Xi_k}\rho)),
\end{aligned} \tag{26}$$

and k-contact k-form $J(\rho)$ on $J^{2r}Y$ by

$$\begin{aligned}
&i_{J^{2r}\Xi_k} \cdots i_{J^{2r}\Xi_2} i_{J^{2r}\Xi_1} J(\rho)\\
&= -\frac{1}{k}(i_{J^{2r}\Xi_k} i_{J^{2r}\Xi_{k-1}} \cdots i_{J^{2r}\Xi_3} i_{J^{2r}\Xi_2} J(i_{J^r\Xi_1}\rho)\\
&- i_{J^{2r}\Xi_k} i_{J^{2r}\Xi_{k-1}} \cdots i_{J^{2r}\Xi_3} i_{J^{2r}\Xi_1} J(i_{J^r\Xi_2}\rho)\\
&- i_{J^{2r}\Xi_k} i_{J^{2r}\Xi_{k-1}} \cdots i_{J^{2r}\Xi_1} i_{J^{2r}\Xi_2} J(i_{J^r\Xi_3}\rho)\\
&- \cdots\\
&- i_{J^{2r}\Xi_k} i_{J^{2r}\Xi_1} \cdots i_{J^{2r}\Xi_1} i_{J^{2r}\Xi_2} J(i_{J^r\Xi_{k-1}}\rho)\\
&- i_{J^{2r}\Xi_1} i_{J^{2r}\Xi_{k-1}} \cdots i_{J^{2r}\Xi_3} i_{J^{2r}\Xi_2} J(i_{J^r\Xi_k}\rho)).
\end{aligned} \tag{27}$$

Note that expressions (22)-(27) express alternations in the corresponding vector fields. For example, let $k = 2$ and let ρ be a 2-contact 3-form on $J^{r+1}Y$,

$$\rho = \sum_{i,j=0}^{r} P^{ij}_{\sigma\nu}\omega^\sigma_i \wedge \omega^\nu_j \wedge dt.$$

Then

$$I(\rho) = \frac{1}{2}\sum_{i,j=0}^{r}\sum_{l=0}^{i}(-1)^i\binom{i}{l}\frac{d^{i-l}}{dt^{i-l}}(P^{ji}_{\nu\sigma} - P^{ij}_{\sigma\nu})\omega^\nu_{j+l} \wedge \omega^\sigma \wedge dt, \tag{28}$$

and

$$J(\rho) = -\frac{1}{2}\sum_{j=0}^{r}\sum_{i=1}^{r}\sum_{l=1}^{i}\sum_{u=0}^{l-1}(-1)^l \binom{l-1}{u}\frac{d^{l-1-u}}{dt^{l-1-u}}(P^{ji}_{\nu\sigma} - P^{ij}_{\sigma\nu})\omega^{\nu}_{j+u} \wedge \omega^{\sigma}_{i-l}. \tag{29}$$

Theorem 1. *Let $k \geq 1$, let ρ be a k-contact $(k+1)$-form on J^rY. Then*

$$(\pi^{2r+1,r})^*\rho = I(\rho) + p_k dJ(\rho). \tag{30}$$

Proof. For $k = 1$, Theorem 1 reduces to the Lemma 1. Now let us suppose that the theorem is fulfilled for some k. We will prove, that (30) holds also for $k+1$. Let ρ be a $(k+1)$-contact $(k+2)$-form on J^rY, let $\Xi_1, \Xi_2, \dots, \Xi_{k+1}$ be a π-vertical vector fields on Y. To compute the contractions in (c) we use the identities

$$i_{J^{2r+1}\Xi}(\pi^{2r+1,r})^*\rho = I(i_{J^r\Xi}\rho) + p_k dJ(i_{J^r\Xi}\rho)$$

and

$$i_{J^{2r+1}\Xi}p_k dJ(\rho) = p_{k-1}di_{J^{2r}\Xi}J(\rho) = p_{k-1}i_{J^{2r}\Xi}dJ(\rho)$$

(we can move $p_k d$ before contraction with cutting of contactness to $p_{k-1}d$, and also inversely).

Then we obtain

$$\begin{aligned}
&i_{J^{2r+1}\Xi_{k+1}} \cdots i_{J^{2r+1}\Xi_2} i_{J^{2r+1}\Xi_1}(\pi^{2r+1,r})^*\rho \\
&= \frac{1}{k+1}(i_{J^{2r+1}\Xi_{k+1}} \cdots i_{J^{2r+1}\Xi_2}(I(i_{J^r\Xi_1}\rho) + p_k dJ(i_{J^r\Xi_1}\rho)) \\
&- i_{J^{2r+1}\Xi_{k+1}} \cdots i_{J^{2r+1}\Xi_1}(I(i_{J^r\Xi_2}\rho) + p_k dJ(i_{J^r\Xi_2}\rho)) \\
&- \cdots \\
&- i_{J^{2r+1}\Xi_1} \cdots i_{J^{2r+1}\Xi_2}(I(i_{J^r\Xi_{k+1}}\rho) + p_k dJ(i_{J^r\Xi_{k+1}}\rho))) \\
&= i_{J^{2r+1}\Xi_{k+1}} \cdots i_{J^{2r+1}\Xi_2} i_{J^{2r+1}\Xi_1} I(\rho) \\
&+ hdi_{J^{2r+1}\Xi_{k+1}} \cdots i_{J^{2r+1}\Xi_2} i_{J^{2r+1}\Xi_1} J(\rho) \\
&= i_{J^{2r+1}\Xi_{k+1}} \cdots i_{J^{2r+1}\Xi_2} i_{J^{2r+1}\Xi_1}(I(\rho) + p_{k+1}dJ(\rho)).
\end{aligned} \tag{31}$$

Omitting contractions we get (30). This completes the proof.

Note that an equivalent formulation of the definition of the operator I can be given by the following formula.

$$\begin{aligned}
&i_{J^{2r+1}\Xi_k} \cdots i_{J^{2r+1}\Xi_2} i_{J^{2r+1}\Xi_1} I(\rho) \\
&= \frac{1}{k}(i_{J^{2r+1}\Xi_k} I(i_{J^r\Xi_{k-1}} \cdots i_{J^r\Xi_3} i_{J^r\Xi_2} i_{J^r\Xi_1}\rho) \\
&- i_{J^{2r+1}\Xi_{k-1}} I(i_{J^r\Xi_k} \cdots i_{J^r\Xi_3} i_{J^r\Xi_2} i_{J^r\Xi_1}\rho) \\
&- \cdots \\
&- i_{J^{2r+1}\Xi_3} I(i_{J^r\Xi_{k-1}} \cdots i_{J^r\Xi_k} i_{J^r\Xi_2} i_{J^r\Xi_1}\rho) \\
&- i_{J^{2r+1}\Xi_2} I(i_{J^r\Xi_{k-1}} \cdots i_{J^r\Xi_3} i_{J^r\Xi_k} i_{J^r\Xi_1}\rho) \\
&- i_{J^{2r+1}\Xi_1} I(i_{J^r\Xi_{k-1}} \cdots i_{J^r\Xi_3} i_{J^r\Xi_2} i_{J^r\Xi_k}\rho)).
\end{aligned} \tag{32}$$

The definition of operator I can be extended to *all* differential $(k+1)$-forms on J^rY (not necessarily k-contact). For an arbitrary differential $(k+1)$-form ρ on J^rY we set

$$\mathcal{I}(\rho) = I(p_k\rho). \tag{33}$$

This gives rise to an $\mathbb{R}$-linear mapping $\mathcal{I} : \Omega^r_{k+1}Y \longrightarrow \Omega^{2r+1}_{k+1}Y$. We call this mapping the *interior Euler-Lagrange operator*.

Main properties of $\mathcal{I}$ can be summarized as follows.

Theorem 2. *Let $\pi : Y \longrightarrow X$ be a fibered manifold over one-dimensional base X. Let $k \geq 0$.*

(a) *For every open set $V \subset Y$ and every $\rho \in \Omega^r_{k+1}V$, $\mathcal{I}(\rho)$ lies in the same class as $(\pi^{2r+1,r})^*\rho$.*

(b) *The operator $\mathcal{I}$ satisfies $\mathcal{I}^2 = \mathcal{I}$ (up to the canonical projection).*

(c) *For every open set $V \subset Y$, the kernels of $\mathcal{I}$ coincide with $\Theta^r_{k+1}V$.*

Remark 1. Anderson [1] defines the interior Euler operator by means of the theory of differential operators on the infinite jet prolongation of the underlying fibered manifold. Similar approach is used by Krbek and Musilová [4] for finite prolongations. On the contrary in our approach only the concept of the Lie derivative on finite order prolongations is used, and the standard rules for computation with Lie derivatives are applied. Formally applying both definitions to a differential form gives the same results.

4. Lepage forms

Let $k \geq 0$. A form $\rho \in \Omega^r_{k+1}V$ is called a *Lepage form*, if

(34) $$p_{k+1}d\rho = \mathcal{I}(d\rho).$$

For $k = 0$, this definition reduces to the original one (Krupka [6], [7]; for more details we refer to Šedĕnková [15]). If ρ is a Lepage form, then the forms $d\rho$ and $\rho + d\eta$, where η is arbitrary, are trivially also Lepage forms. The meaning of Lepage forms consists in a generalization of formulas (9), (10); if $k = 1$, then $p_2 d\rho$ is the *Helmholtz form* (compare with Krupka [8] for the first order case).

We now analyze the structure of Lepage 2-forms in higher order mechanics.

Theorem 3. *Let $V \subset Y$ be an open set. Let $\rho \in \Omega^1_2 V$, let in a fibered chart*

(35) $$\rho = a_\sigma \omega^\sigma \wedge dt + b_\sigma d\dot{q}^\sigma \wedge dt + c_{\sigma\nu}\omega^\sigma \wedge \omega^\nu + d_{\sigma\nu} d\dot{q}^\sigma \wedge \omega^\nu + e_{\sigma\nu} d\dot{q}^\sigma \wedge d\dot{q}^\nu,$$

where the coefficients $c_{\sigma\nu}$, $e_{\sigma\nu}$ are antisymmetric in σ, ν. The following three conditions are equivalent:

(a) *ρ is a Lepage form.*

(b) *ρ satisfies*

(36) $$0 = \frac{\partial e_{\sigma\nu}}{\partial \dot{q}^\lambda} + \frac{\partial e_{\nu\lambda}}{\partial \dot{q}^\sigma} + \frac{\partial e_{\lambda\sigma}}{\partial \dot{q}^\nu},$$

(37) $$0 = d_{\sigma\nu} - d_{\nu\sigma} - \frac{\partial b_\sigma}{\partial \dot{q}^\nu} + \frac{\partial b_\nu}{\partial \dot{q}^\sigma} + 2\frac{\partial e_{\sigma\nu}}{\partial t} + 2\frac{\partial e_{\sigma\nu}}{\partial q^\lambda}\dot{q}^\lambda,$$

(38) $$0 = \frac{\partial d_{\sigma\nu}}{\partial \dot{q}^\lambda} - \frac{\partial d_{\nu\sigma}}{\partial \dot{q}^\lambda} + \frac{\partial d_{\lambda\sigma}}{\partial \dot{q}^\nu} - \frac{\partial d_{\lambda\nu}}{\partial \dot{q}^\sigma} + 2\frac{\partial e_{\lambda\sigma}}{\partial q^\nu} - 2\frac{\partial e_{\lambda\nu}}{\partial q^\sigma},$$

(39) $$\begin{aligned} 0 = {} & \frac{\partial a_\nu}{\partial \dot{q}^\sigma} - \frac{\partial a_\sigma}{\partial \dot{q}^\nu} + \frac{\partial b_\nu}{\partial q^\sigma} - \frac{\partial b_\sigma}{\partial q^\nu} + \frac{\partial d_{\sigma\nu}}{\partial t} - \frac{\partial d_{\nu\sigma}}{\partial t} \\ & + \frac{\partial d_{\sigma\nu}}{\partial q^\lambda}\dot{q}^\lambda - \frac{\partial d_{\nu\sigma}}{\partial q^\lambda}\dot{q}^\lambda + 4c_{\sigma\nu}. \end{aligned}$$

(c) *There exist functions* A_σ*, and a 1-form* η *such that*

$$\rho = A_\sigma \omega^\sigma \wedge dt + \frac{1}{4}\left(\frac{\partial A_\sigma}{\partial \dot{q}^\nu} - \frac{\partial A_\nu}{\partial \dot{q}^\sigma}\right)\omega^\sigma \wedge \omega^\nu + d\eta. \tag{40}$$

Now we consider second order 2-forms

$$\begin{aligned}\rho = {} & a_\sigma \omega^\sigma \wedge dt + b_\sigma \dot{\omega}^\sigma \wedge dt + c_\sigma d\ddot{q}^\sigma \wedge dt + d_{\sigma\nu}\omega^\sigma \wedge \omega^\nu + e_{\sigma\nu}\dot{\omega}^\sigma \wedge \omega^\nu \\ & + f_{\sigma\nu}\dot{\omega}^\sigma \wedge \dot{\omega}^\nu + g_{\sigma\nu} d\ddot{q}^\sigma \wedge \omega^\nu + h_{\sigma\nu} d\ddot{q}^\sigma \wedge \dot{\omega}^\nu + i_{\sigma\nu} d\ddot{q}^\sigma \wedge d\ddot{q}^\nu,\end{aligned} \tag{41}$$

where the coefficients $d_{\sigma\nu}$, $f_{\sigma\nu}$, $i_{\sigma\nu}$ are antisymmetric in σ, ν. One can easily see that

$$\begin{aligned}p_2 d\rho = {} & P_{\sigma\nu}\omega^\sigma \wedge \omega^\nu \wedge dt + Q_{\sigma\nu}\dot{\omega}^\sigma \wedge \omega^\nu \wedge dt + R_{\sigma\nu}\ddot{\omega}^\sigma \wedge \omega^\nu \wedge dt \\ & + S_{\sigma\nu}\dot{\omega}^\sigma \wedge \dot{\omega}^\nu \wedge dt + T_{\sigma\nu}\ddot{\omega}^\sigma \wedge \dot{\omega}^\nu \wedge dt + U_{\sigma\nu}\ddot{\omega}^\sigma \wedge \ddot{\omega}^\nu \wedge dt,\end{aligned} \tag{42}$$

where

$$\begin{aligned}
P_{\sigma\nu} &= \frac{1}{2}\left(\frac{\partial a_\nu}{\partial q^\sigma} - \frac{\partial a_\sigma}{\partial q^\nu} + 2\frac{d}{dt}d_{\sigma\nu} + \left(\frac{\partial g_{\lambda\sigma}}{\partial q^\nu} - \frac{\partial g_{\lambda\nu}}{\partial q^\sigma}\right)\dddot{q}^\lambda\right),\\
Q_{\sigma\nu} &= \frac{\partial a_\nu}{\partial \dot{q}^\sigma} - \frac{\partial b_\sigma}{\partial q^\nu} + 2d_{\sigma\nu} + \frac{d}{dt}e_{\sigma\nu} - \left(\frac{\partial g_{\lambda\nu}}{\partial \dot{q}^\sigma} - \frac{\partial h_{\lambda\sigma}}{\partial q^\nu}\right)\dddot{q}^\lambda,\\
R_{\sigma\nu} &= \frac{\partial a_\nu}{\partial \ddot{q}^\sigma} - \frac{\partial c_\sigma}{\partial q^\nu} + e_{\sigma\nu} + \frac{d}{dt}g_{\sigma\nu} - \left(\frac{\partial g_{\lambda\nu}}{\partial \ddot{q}^\sigma} - 2\frac{\partial i_{\lambda\sigma}}{\partial q^\nu}\right)\dddot{q}^\lambda,\\
S_{\sigma\nu} &= \frac{1}{2}\left(\frac{\partial b_\nu}{\partial \dot{q}^\sigma} - \frac{\partial b_\sigma}{\partial \dot{q}^\nu} + e_{\sigma\nu} - e_{\nu\sigma} + 2\frac{d}{dt}f_{\sigma\nu} + \left(\frac{\partial h_{\lambda\sigma}}{\partial \dot{q}^\nu} - \frac{\partial h_{\lambda\nu}}{\partial \dot{q}^\sigma}\right)\dddot{q}^\lambda\right),\\
T_{\sigma\nu} &= \frac{\partial b_\nu}{\partial \ddot{q}^\sigma} - \frac{\partial c_\sigma}{\partial \dot{q}^\nu} + 2f_{\sigma\nu} + g_{\sigma\nu} + \frac{d}{dt}h_{\sigma\nu} - \left(\frac{\partial h_{\lambda\nu}}{\partial \ddot{q}^\sigma} - 2\frac{\partial i_{\lambda\sigma}}{\partial \dot{q}^\nu}\right)\dddot{q}^\lambda,\\
U_{\sigma\nu} &= \frac{1}{2}\left(\frac{\partial c_\nu}{\partial \ddot{q}^\sigma} - \frac{\partial c_\sigma}{\partial \ddot{q}^\nu} + h_{\sigma\nu} - h_{\nu\sigma} + 2\frac{d}{dt}i_{\sigma\nu} + 2\left(\frac{\partial i_{\lambda\sigma}}{\partial \ddot{q}^\nu} - \frac{\partial i_{\lambda\nu}}{\partial \ddot{q}^\sigma}\right)\dddot{q}^\lambda\right).
\end{aligned} \tag{43}$$

We have the following result.

Theorem 4. *Let* $\rho \in \Omega_2^2 V$ *be a form expressed by* (41)*. The following three conditions are equivalent:*

(a) ρ *is a Lepage form.*

(b) *The components of* $p_2 d\rho$ *satisfy*

$$\begin{aligned} &U_{\sigma\nu} - U_{\nu\sigma} = 0, \qquad T_{\sigma\nu} = 0, \qquad S_{\sigma\nu} - S_{\nu\sigma} = 0,\\ &R_{\sigma\nu} + R_{\nu\sigma} = 0, \qquad Q_{\sigma\nu} - Q_{\nu\sigma} - 2\frac{d}{dt}R_{\sigma\nu} = 0.\end{aligned} \tag{44}$$

(c) *There exist functions* A_σ *satisfying*

$$\frac{\partial}{\partial \ddot{q}^\tau}\left(\frac{\partial A_\sigma}{\partial \ddot{q}^\nu} - \frac{\partial A_\nu}{\partial \ddot{q}^\sigma}\right) = 0 \tag{45}$$

and a 1-form η *such that*

$$\begin{aligned}\rho = {} & A_\sigma \omega^\sigma \wedge dt + \frac{1}{4}\left(\frac{\partial A_\sigma}{\partial \dot{q}^\nu} - \frac{\partial A_\nu}{\partial \dot{q}^\sigma} - \frac{d}{dt}\left(\frac{\partial A_\sigma}{\partial \ddot{q}^\nu} - \frac{\partial A_\nu}{\partial \ddot{q}^\sigma}\right)\right)\omega^\sigma \wedge \omega^\nu \\ & - \frac{1}{2}\left(\frac{\partial A_\sigma}{\partial \ddot{q}^\nu} + \frac{\partial A_\nu}{\partial \ddot{q}^\sigma}\right)\dot{\omega}^\sigma \wedge \omega^\nu + d\eta.\end{aligned} \tag{46}$$

Proof. Theorem 4 can be verified by a direct computation.

We note that the coefficients in (42) can be expressed in terms of the coefficients in (41), then (44) become conditions for the coefficients of ρ.

Analogous results can also be given for r-th order 2-forms

$$\rho = \sum_{i=0}^{r-1} a_\sigma^i \omega_i^\sigma \wedge dt + b_\sigma^r dq_r^\sigma \wedge dt + \sum_{i,j=0}^{r-1} c_{\sigma\nu}^{ij} \omega_i^\sigma \wedge \omega_j^\nu + \sum_{j=0}^{r-1} d_{\sigma\nu}^{rj} dq_r^\sigma \wedge \omega_j^\nu + e_{\sigma\nu}^{rr} dq_r^\sigma \wedge dq_r^\nu, \tag{47}$$

where the coefficients $e_{\sigma\nu}^{rr}$ are antisymmetric in σ, ν, the coefficients $c_{\sigma\nu}^{ij}$ are antisymmetric in pairs $\binom{i}{\sigma}$, $\binom{j}{\nu}$. One can easily see that

$$p_2 d\rho = \sum_{j=0}^{r} H_{\sigma\nu}^{0j} \omega^\sigma \wedge \omega_j^\nu \wedge dt + \sum_{i,j=1}^{r} H_{\sigma\nu}^{ij} \omega_i^\sigma \wedge \omega_j^\nu \wedge dt, \tag{48}$$

where

$$H_{\sigma\nu}^{00} = \frac{1}{2}\left(\frac{\partial a_\nu^0}{\partial q_0^\sigma} - \frac{\partial a_\sigma^0}{\partial q_0^\nu} + 2\frac{d}{dt}c_{\sigma\nu}^{00} + \frac{\partial d_{\lambda\sigma}^{r0}}{\partial q_0^\nu} - \frac{\partial d_{\lambda\nu}^{r0}}{\partial q_0^\sigma}\right),$$

$$H_{\sigma\nu}^{0j} = -\frac{\partial a_\sigma^0}{\partial q_j^\nu} + \frac{\partial a_\nu^j}{\partial q_0^\sigma} + 2\frac{d}{dt}c_{\sigma\nu}^{0j} + c_{\sigma\nu}^{0\,j-1} - c_{\nu\ \sigma}^{j-1\,0} + \left(\frac{\partial d_{\lambda\sigma}^{r0}}{\partial q_j^\lambda} - \frac{\partial d_{\lambda\nu}^{rj}}{\partial q_0^\sigma}\right) q_{r+1}^\lambda,$$

$$H_{\sigma\nu}^{0r} = \frac{\partial a_\sigma^0}{\partial q_r^\nu} + \frac{\partial b_\nu^r}{\partial q_0^\sigma} + c_{\sigma\nu}^{0\,r-1} - c_{\nu\ \sigma}^{r-1\,0} - \frac{d}{dt}d_{\nu\sigma}^{r0} + \left(\frac{\partial d_{\lambda\sigma}^{r0}}{\partial q_r^\nu} + 2\frac{\partial e_{\nu\lambda}^{rr}}{\partial q_0^\sigma}\right) q_{r+1}^\lambda,$$

$$H_{\sigma\nu}^{ij} = \frac{1}{2}\left(\frac{\partial a_\nu^j}{\partial q_i^\sigma} - \frac{\partial a_\sigma^i}{\partial q_j^\nu} + 2\frac{d}{dt}c_{\sigma\nu}^{ij} + 2(c_{\sigma\ \nu}^{i-1\,j} - c_{\nu\ \sigma}^{j-1\,i}) + \left(\frac{\partial d_{\lambda\sigma}^{ri}}{\partial q_j^\nu} - \frac{\partial d_{\lambda\nu}^{rj}}{\partial q_i^\sigma}\right) q_{r+1}^\lambda\right),$$

$$H_{\sigma\nu}^{ir} = \frac{1}{2}\Bigg(-\frac{\partial a_\sigma^i}{\partial q_r^\nu} + \frac{\partial a_\nu^r}{\partial q_i^\sigma} + \frac{\partial b_\nu^r}{\partial q_i^\sigma} - \frac{\partial b_\sigma^i}{\partial q_r^\nu} + 2(c_{\sigma\nu}^{i\,r-1} - c_{\nu\sigma}^{r\,i-1}) - \frac{d}{dt}(d_{\nu\sigma}^{ri} - d_{\sigma\nu}^{ir}) - d_{\nu\sigma}^{r\,i-1} + d_{\sigma\nu}^{i\,r-1} + \left(\frac{\partial d_{\lambda\sigma}^{ri}}{\partial q_r^\nu} - \frac{\partial d_{\lambda\nu}^{rr}}{\partial q_i^\sigma} + \frac{\partial e_{\nu\lambda}^{rr}}{\partial q_i^\sigma} - \frac{\partial e_{\sigma\lambda}^{ir}}{\partial q_r^\nu}\right) q_{r+1}^\lambda$$

$$H_{\sigma\nu}^{rr} = \frac{1}{2}\left(\frac{\partial b_\nu^r}{\partial q_r^\sigma} - \frac{\partial b_\sigma^r}{\partial q_r^\nu} + d_{\sigma\nu}^{r\,r-1} - d_{\nu\sigma}^{r\,r-1} + 2\frac{d}{dt}e_{\sigma\nu}^{rr} + \frac{\partial e_{\nu\lambda}^{rr}}{\partial q_r^\sigma} - \frac{\partial e_{\sigma\lambda}^{rr}}{\partial q_r^\nu}\right).$$

Theorem 5. *Let $\rho \in \Omega_2^r V$ be a form expressed by* (47)*. Then ρ is Lepage if and only if*

$$H_{\sigma\nu}^{ij} - H_{\nu\sigma}^{ji} = 0, \qquad 1 \le i, j \le r, \tag{49}$$

$$H_{\sigma\nu}^{0j} + (-1)^j H_{\nu\sigma}^{0j} + \sum_{l=j+1}^{r} (-1)^l \binom{l}{j} \frac{d^{l-j}}{dt^{l-j}} H_{\nu\sigma}^{0l} = 0, \qquad 1 \le j \le r. \tag{50}$$

Proof. Theorem 5 can be verified by a direct computation.

Finally, we define Lepage equivalents of the canonical representatives of differential forms. Let $\beta \in \Omega^s_{k+1}/\Theta^s_{k+1}$ be a class, i.e., let $\beta = \mathcal{I}(\eta)$ for some $\eta \in \Omega^s_{k+1}V$. A form $\rho \in \Omega^r_{k+1}V$ is said to be a *Lepage equivalent* of β, if ρ is a Lepage form, and

$$p_k\rho = \beta. \tag{51}$$

In particular, this definition includes Lepage equivalents of *dynamical forms* (i.e., the canonical representatives of 2-forms). Let

$$\varepsilon = \varepsilon_\sigma \omega^\sigma \wedge dt$$

be a second order dynamical form. Then Lepage equivalent ρ_ε of the dynamical form ε has the form

$$\begin{aligned}\rho_\varepsilon = \varepsilon_\sigma \omega^\sigma \wedge dt &+ \frac{1}{4}\left(\frac{\partial \varepsilon_\sigma}{\partial \dot{q}^\nu} - \frac{\partial \varepsilon_\nu}{\partial \dot{q}^\sigma} - \frac{d}{dt}\left(\frac{\partial \varepsilon_\sigma}{\partial \ddot{q}^\nu} - \frac{\partial \varepsilon_\nu}{\partial \ddot{q}^\sigma}\right)\right)\omega^\sigma \wedge \omega^\nu \\ &- \frac{1}{2}\left(\frac{\partial \varepsilon_\sigma}{\partial \ddot{q}^\nu} + \frac{\partial \varepsilon_\nu}{\partial \ddot{q}^\sigma}\right)\dot{\omega}^\sigma \wedge \omega^\nu\end{aligned} \tag{52}$$

(compare with second order Lepage form (46)).

The Helmholtz form of ε is the class $[d\rho_\varepsilon] = H_\varepsilon = p_2 d\rho_\varepsilon$. Explicitly,

$$\begin{aligned}H_\varepsilon = \frac{1}{2}&\left(\frac{\partial \varepsilon_\sigma}{\partial q^\nu} - \frac{\partial \varepsilon_\nu}{\partial q^\sigma} - \frac{1}{2}\frac{d}{dt}\left(\frac{\partial \varepsilon_\sigma}{\partial \dot{q}^\nu} - \frac{\partial \varepsilon_\nu}{\partial \dot{q}^\sigma}\right)\right. \\ &+ \left.\frac{1}{2}\frac{d^2}{dt^2}\left(\frac{\partial \varepsilon_\sigma}{\partial \ddot{q}^\nu} - \frac{\partial \varepsilon_\nu}{\partial \ddot{q}^\sigma}\right)\right)\omega^\nu \wedge \omega^\sigma \wedge dt \\ &+ \left(\frac{1}{2}\left(\frac{\partial \varepsilon_\sigma}{\partial \dot{q}^\nu} + \frac{\partial \varepsilon_\nu}{\partial \dot{q}^\sigma}\right) - \frac{d}{dt}\frac{\partial \varepsilon_\nu}{\partial \ddot{q}^\sigma}\right)\dot{\omega}^\nu \wedge \omega^\sigma \wedge dt \\ &+ \frac{1}{2}\left(\frac{\partial \varepsilon_\sigma}{\partial \ddot{q}^\nu} - \frac{\partial \varepsilon_\nu}{\partial \ddot{q}^\sigma}\right)\ddot{\omega}^\nu \wedge \omega^\sigma \wedge dt.\end{aligned}$$

It is known from the variational sequence that ε is variational if and only if $H_\varepsilon = 0$. The Helmholtz form one can find also in [5].

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