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Variational principles for locally variational forms

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Abstract. We present the theory of higher order local variational principles in fibered manifolds, in which the fundamental global concept is a locally variational dynamical form. Any two Lepage forms, defining a local variational principle for this form, differ on intersection of their domains, by a variationally trivial form. In this sense, but in a different geometric setting, the local variational principles satisfy analogous properties as the variational functionals of the Chern-Simons type. The resulting theory of extremals and symmetries extends the firts order theories of the Lagrange-Souriau form, presented by Grigore and Popp, and closed equivalents of the first order Euler-Lagrange forms of Haková and Krupková. Conceptually, our approach differs from Prieto, who uses the Poincaré-Cartan forms, which do not have higher order global analogues.

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1. Introduction

It is well known that differential equations for critical points of a variational functional in a fibered manifold can be represented by a global differential form, the *Euler-Lagrange form*, whose components are the Euler-Lagrange expressions. It is also well known that there exist differential equations, represented by similar global differential forms, the *dynamical forms*, which are *locally variational*, but do not admit a *global* lagrangian. A deeper understanding of this phenomenon is provided by the variational bicomplex theory (Vinogradov [31], Takens [28], Anderson and Duchamp [2], Dedecker and Tulczyjew [6], Tulczyjew [30]), and the (finite order) variational sequence theory (Krupka [20], Grigore [12], Vitolo [32], Krbek and Musilová [14]).

The corresponding variational principles in the first order field theory have been recently studied by several authors. Grigore and Popp [11] extended the ideas of Souriau [27] on the role of closed 2-forms in mechanics to $(n+1)$-forms in the variational theory for

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n-dimensional submanifolds of a given manifold. They introduced the *Lagrange-Souriau form*, representing the Euler-Lagrange equations, and proved that this form is equal to the exterior derivative of the *fundamental Lepage form* in the sense of Krupka [19], [16] (see also Betounes [3], [4], and Rund [25]). The theory presented by Prieto [23], [24]), is based on the existence of the global Poincaré-Cartan form (Sniatycki [26], Goldschmidt and Sternberg [9], Krupka [15], [19], García [8]), and is aimed to extend basic properties of variational principles of the Chern-Simons type (see e.g. Freed [7]) to fibered manifolds. Haková and Krupková [13] showed that the closed $(n+1)$-forms related to variational systems of first order partial differential equations are exactly the exterior derivative of the fundamental Lepage form.

Closed 2-forms in higher order mechanics, equivalent with the Euler-Lagrange forms, were studied by Krupková [21], [22].

This paper is devoted to local variationality in the framework of the *higher order* variational theory on fibered spaces (Krupka [17], [19]), and the variational sequence theory. In general, for higher order lagrangians in field theory a global analogue of the Poincaré-Cartan form does not exist. We show that instead of this form one can use *any* Lepage form; the Poincaré-Cartan form is an example of a first order Lepage form. Any (higher order) Lepage form gives rise, by means of the global variation formula, to the (higher order) *Euler-Lagrange form.* Conceptually, the theory is quite simple and clear. In particular, it is easy to understand, in full generality, that there exist (global) dynamical forms, admitting local higher order lagrangians, but not a global one.

In Section 2 we give a survey of the higher order variational theory on fibered spaces. Section 3 is devoted to some new results on infinitesimal symmetries, based on the fundamental Lepage form. In Section 4 we introduce a *local variational principle* for a *locally variational* dynamical form. We give the *first variation formula* and discuss properties of transformations, leaving invariant the local variational principle, and the locally variational form.

In this paper we suppose that we have a fibered manifold $\pi : Y \longrightarrow X$, and write $n = \dim X$, and $n + m = \dim Y$. J^rY is the *r-jet prolongation* of Y, and $\pi^{r,s} : J^rY \longrightarrow J^sY$, $\pi^r : J^rY \longrightarrow X$ are the *canonical jet projections.* The *r-jet prolongation* of a section γ is defined to be the mapping $x \longrightarrow J^r\gamma(x) = J^r_x\gamma$. For any set $W \subset Y$ we denote $W^r = (\pi^{r,0})^{-1}(W)$. Any *fibered chart* (V,ψ), $\psi = (x^i, y^\sigma)$, on Y, induces the *associated charts* on X and on J^rY, denoted by (U,φ), $\varphi = (x^i)$, and (V^r,ψ^r), $\psi^r = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1j_2}, \dots, y^\sigma_{j_1j_2\dots j_r})$, respectively; here $1 \le i \le n$, $1 \le \sigma \le m$, and $V^r = (\pi^{r,0})^{-1}(V)$, $U = \pi^r(V)$. We denote $\omega_0 = dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$, and

$$\omega_k = i_{\partial/\partial x^k}\omega_0 = (-1)^{k-1} dx^1 \wedge dx^2 \wedge \ldots \wedge dx^{k-1} \wedge dx^{k+1} \wedge \ldots \wedge dx^n.$$

We define the *formal derivative operator* by

$$d_i = \frac{\partial}{\partial x^i} + y^\sigma_i \frac{\partial}{\partial y^\sigma} + y^\sigma_{i_1 i} \frac{\partial}{\partial y^\sigma_{i_1}} + \ldots + y^\sigma_{i_1 i_2 \dots i_r i} \frac{\partial}{\partial y^\sigma_{i_1 i_2 \dots i_r}}.$$

2. Lagrange structures

2.1. Differential forms on jet spaces

For any open set $W \subset Y$, let $\Omega^r_0 W$ be the ring of functions on W^r. The $\Omega^r_0 W$-module of differential q-forms on W^r is denoted by $\Omega^r_q W$, and the exterior algebra of forms on W^r is denoted by $\Omega^r W$. The module of $\pi^{r,0}$-horizontal (π^r-horizontal) q-forms is denoted by $\Omega^r_{q,Y} W$ ($\Omega^r_{q,X} W$, respectively); forms belonging to these spaces are sometimes called $\pi^{r,0}$-semibasic, or π^r-semibasic, respectively.

Let $W \subset Y$ be an open set. The fibered structure of Y induces a morphism of exterior algebras $h : \Omega^r W \longrightarrow \Omega^{r+1} W$, called the *horizontalization*. In a fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, h is defined by

$$hf = f \circ \pi^{r+1,r}, \quad hdx^i = dx^i, \quad hdy^\sigma_{j_1 j_2 \dots j_p} = y^\sigma_{j_1 j_2 \dots j_p k} dx^k,$$

where $f : W^r \longrightarrow \mathbb{R}$ is a function, and $0 \leq p \leq r$. Note that h can be defined intrinsically: for a k-form $\eta \in \Omega^r_k W$, where $0 \leq k \leq n$, we define $h\eta$ to be a unique π^{r+1}-horizontal form such that $J^r\gamma^*\eta = J^{r+1}\gamma^* h\eta$ for every section γ of Y (here * denotes the pull-back operation).

We say that a form $\eta \in \Omega^r_k W$ is *contact*, if $h\eta = 0$. For any fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, the 1-forms

$$\omega^\sigma_{j_1 j_2 \dots j_p} = dy^\sigma_{j_1 j_2 \dots j_p} - y^\sigma_{j_1 j_2 \dots j_p k} dx^k,$$

where $1 \leq p \leq r - 1$, are examples of contact 1-forms. Note that these forms define a basis of 1-forms on V^r, $(dx^i, \omega^\sigma_{j_1 j_2 \dots j_p}, dy^\sigma_{j_1 j_2 \dots j_r})$.

It is known that a form $\eta \in \Omega^r_k W$ has a unique decomposition

$$(1) \qquad (\pi^{r+1,r})^*\eta = h\eta + p_1\eta + p_2\eta + \dots + p_k\eta,$$

such that $p_i\eta$ contains, in any fibered chart, exactly i exterior factors $\omega^\sigma_{j_1 j_2 \dots j_l}$, $1 \leq l \leq r$. In particular, this gives us a simple formulation of the fact that the forms $\omega^\sigma_{j_1 j_2 \dots j_l}$ generate an *ideal* in the exterior algebra $\Omega^r V$ (the *contact ideal*).

$h\eta$ ($p_i\eta$) is the *horizontal* (*i-th contact*) *component* of η. The decomposition (1) is *invariant*, and is called the *canonical decomposition* of η.

η is π^r*-horizontal* if and only if $(\pi^{r+1,r})^*\eta = h\eta$. We say that η is k*-contact*, if $(\pi^{r+1,r})^*\eta = p_k\eta$; in this case k is the *order of contactness* of η.

Let $k \geq n + 1$. Then for any k-form $\eta \in \Omega^r_k W$, $h\eta = 0$, $p_1\eta = 0$, $p_2\eta = 0$, $\dots$, $p_{k-n-1}\eta = 0$, because every of these forms contains more than n exterior factors dx^i. η is said to be *strongly contact*, if $p_{k-n}\eta = 0$.

2.2. Lagrangians

A *lagrangian* (of *order* r) for Y is any π^r-horizontal n-form on some $W^r \subset J^rY$, i.e., any element of the set $\Omega^r_{n,X} W$. In a fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, a lagrangian of order r defined on $V^r = (\pi^{r,0})^{-1}(V)$ has an expression

$$(2) \qquad \lambda = \mathcal{L}\omega_0,$$

where $\mathcal{L} : V^r \longrightarrow \mathbb{R}$ is a function (the *Lagrange function* associated with λ and (V, ψ)). Clearly, in general a lagrangian cannot be determined by a *globally* defined function unless a volume element on X is specified.

A pair (Y, λ), consisting of a fibered manifold Y and a lagrangian λ of order r for Y is called a *Lagrange structure* (of order r).

Sometimes it is convenient to use lagrangians of the form $\lambda = h\eta$, where $\eta \in \Omega^{r-1}_n W$. These lagrangians have a certain polynomial structure in the highest order variables $y^\sigma_{j_1 j_2 \dots j_r}$. The assumption $\lambda = h\eta$ appears naturally in the variational sequence theory, but does not restrict the generality.

Note that our definition includes lagrangians defined over any open subsets $W \subset Y$; we need such a definition to describe phenomena arising in connection with the so called *local variational principles* for *globally defined* Euler-Lagrange equations. The discussion of this situation is a main objective of this paper.

2.3. Lepage forms

We now give a formal definition of a Lepage form (Krupka [19]). A principal geometric meaning of this concept consists in the fact, that Lepage forms describe the relationship between the equations for extremals of variational principles on one side, and the exterior derivative operator, acting on differential forms, on the other side.

A differential form $\rho \in \Omega_n^s W$, where $n = \dim X$, is called a *Lepage form*, if $p_1 d\rho$ is $\pi^{s+1,0}$-horizontal, i.e., $p_1 d\rho \in \Omega_{n+1,Y}^{s+1} W$. A Lepage form ρ is a *Lepage equivalent* of a lagrangian $\lambda \in \Omega_{n,X}^r W$, if the horizontal component of ρ coincides with λ, i.e., $h\rho = \lambda$ (possibly up to a jet projection).

If ρ is a Lepage equivalent of a lagrangian $\lambda \in \Omega_{n,X}^r W$, expressed by (2), then one can get by a direct calculation

(3) $$p_1 d\rho = E_\sigma(\mathcal{L})\omega^\sigma \wedge \omega_0,$$

where

(4) $$E_\sigma(\mathcal{L}) = \sum_{k=0}^{r} (-1)^k d_{i_1} d_{i_2} \dots d_{i_k} \frac{\partial \mathcal{L}}{\partial y^\sigma_{i_1 i_2 \dots i_k}}$$

are the *Euler-Lagrange expressions* associated with the Lagrange function $\mathcal{L}$. In particular, $p_1 d\rho$ depends on the lagrangian λ only. The $(n+1)$-form

$$E_\lambda = p_1 d\rho$$

is called the *Euler-Lagrange form* associated with λ.

We give three examples of Lepage equivalents:

(1) Every first order lagrangian $\lambda \in \Omega_{n,X}^1 W$ has a unique Lepage equivalent $\Theta_\lambda \in \Omega_{n,Y}^1 W$ whose order of contactness is ≤ 1. If λ is expressed in a fibered chart by $\lambda = \mathcal{L}\omega_0$, then

$$\Theta_\lambda = \mathcal{L}\omega_0 + \frac{\partial \mathcal{L}}{\partial y^\sigma_i}\omega^\sigma \wedge \omega_i.$$

Θ_λ is the *Poincaré-Cartan equivalent* of λ, or the *Poincaré-Cartan form.*

(2) Let $\lambda \in \Omega_{n,X}^1 W$ be as above. The *fundamental Lepage equivalent* $\Phi_\lambda \in \Omega_{n,Y}^1 W$ of λ is given by

(5) $$\Phi_\lambda = \sum_{k=0}^{n} \left(\frac{1}{k!}\right)^2 \frac{\partial^k \mathcal{L}}{\partial y^{\sigma_1}_{j_1} \partial y^{\sigma_2}_{j_2} \dots \partial y^{\sigma_k}_{j_k}} \omega^{\sigma_1} \wedge \omega^{\sigma_2} \wedge \dots \wedge \omega^{\sigma_k} \wedge \omega_{j_1 j_2 \dots j_k},$$

where

$$i_{\partial/\partial x^{i_k}} \dots i_{\partial/\partial x^{i_2}} i_{\partial/\partial x^{i_1}} \omega_0 = \omega_{i_1 i_2 \dots i_k}.$$

Φ_λ has the following remarkable properties: (a) $d\Phi_\lambda = 0$ if and only if $E_\lambda = 0$, and (b) $\lambda = h\eta$ for some $\eta \in \Omega_n^0 W$ if and only if E_λ is $\pi^{2,1}$-projectable. The form Φ_λ was introduced for the first time by Krupka ([19], [16]), and it was rediscovered by Betounes [3], [4], and Rund [25] who wrote Φ_λ in a more simple way as it stands in (5).

(3) Expression

(6) $$\Theta_\lambda = \mathcal{L}\omega_0 + \left(\frac{\partial \mathcal{L}}{\partial y^\sigma_i} - d_p \frac{\partial \mathcal{L}}{\partial y^\sigma_{pi}}\right)\omega^\sigma \wedge \omega_i + \frac{\partial \mathcal{L}}{\partial y^\sigma_{ji}}\omega^\sigma_j \wedge \omega_i$$

generalizes the Poincaré-Cartan form to *second order* lagrangians $\lambda \in \Omega_{n,X}^2 W$ (Krupka [19]), higher order generalizations can be found in Krupka [17]. It can be shown that every Lepage equivalent of a lagrangian $\lambda = \mathcal{L}\omega_0$ of order r has the chart expression

$\rho = \Theta_\lambda + d\mu + \nu$, where

$$\Theta_\lambda = \mathcal{L}\omega_0 + \sum_{k=0}^{s}\left(\sum_{l=0}^{r-k}(-1)^l d_{i_1}d_{i_2}\dots d_{i_l}\frac{\partial\mathcal{L}}{\partial y^\sigma_{i_1i_2\dots i_lj_1j_2\dots j_k}}\right)\omega^\sigma_{j_1j_2\dots j_k}\wedge\omega_i, \tag{7}$$

μ is a contact form, and ν is of order of contactness ≥ 2. Expression (6) defines a differential form on J^3Y, but for $r \geq 3$, the (local) Lepage equivalents (7) of λ are no longer invariant.

2.4. Automorphisms, variations

By an *automorphism* of Y we mean a diffeomorphism $\alpha : W \longrightarrow Y$, where $W \subset Y$ is an open set, such that there exists a diffeomorphism $\alpha_0 : \pi(W) \longrightarrow X$ such that $\pi\alpha = \alpha_0\pi$. If α_0 exists, it is unique, and is called the *π-projection* of α. The *r-jet prolongation* of α is an automorphism $J^r\alpha : W^r \longrightarrow J^rY$ of J^rY, defined by

$$J^r\alpha(J^r_x\gamma) = J^r_{\alpha_0(x)}(\alpha\gamma\alpha_0^{-1}).$$

Let $U \subset X$ be an open set, and let $\gamma : U \longrightarrow Y$ be a section. Let ξ be a π-projectable vector field on an open set $W \subset Y$ such that $\gamma(U) \subset W$. If α_t is the local one-parameter group of ξ, and $\alpha_{(0)t}$ is its projection, then since $\pi\alpha_t = \alpha_{(0)t}\pi$,

$$\gamma_t = \alpha_t\gamma\alpha^{-1}_{(0)t}$$

is one-parameter family of sections of Y, depending smoothly on t. γ_t is called the *variation*, or the *deformation* of γ, *induced* by ξ.

We define the *r-jet prolongation* of ξ to be the vector field $J^r\xi$ on J^rY whose local one-parameter group is $J^r_{\alpha_t}$. Thus,

$$J^r\xi(J^r_x\gamma) = \left\{\frac{d}{dt}J^r_{\alpha_{(0)t}(x)}(\alpha_t\gamma\alpha^{-1}_{(0)t})\right\}_0.$$

2.5. Global variational functionals

Let Ω be a piece of X (a compact, n-dimensional submanifold of X with boundary $\partial\Omega$), let $\Gamma_{\Omega,W}(\pi)$ be the set of smooth sections γ over Ω such that $\gamma(\Omega) \subset W$. Suppose that we have a lagrangian $\lambda \in \Omega^r_{n,X}(W)$. This gives rise to the *variational functional*, or the *action function* associated with λ, $\Gamma_{\Omega,W}(\pi) \ni \gamma \longrightarrow \lambda_\Omega(\gamma) \in \mathbb{R}$, defined by

$$\lambda_\Omega(\gamma) = \int_\Omega J^r\gamma^*\lambda.$$

Choose a section $\gamma \in \Gamma_{\Omega,W}(\pi)$ and a π-projectable vector field ξ on Y, and consider the induced variation γ_t of γ. Since the domain of γ_t contains Ω for all sufficiently small t, we get a real-valued function on a neighborhood $(-\epsilon, \epsilon)$ of the origin $0 \in \mathbb{R}$,

$$(-\epsilon,\epsilon) \ni t \longrightarrow \lambda_{\alpha_{(0)t}(\Omega)}(\alpha_t\gamma\alpha^{-1}_{(0)t}) = \int_{\alpha_{(0)t}(\Omega)} J^r(\alpha_t\gamma\alpha^{-1}_{(0)t})^*\lambda \in \mathbb{R}.$$

Differentiating this function at $t = 0$ we obtain

$$(\partial_{J^r\xi}\lambda)_\Omega(\gamma) = \int_\Omega J^r\gamma^*\partial_{J^r\xi}\lambda, \tag{8}$$

where $\partial_{J^r\xi}\lambda$ is the Lie derivative of λ by $J^r\xi$. The number (8) is the *variation* of the variational function λ_Ω at γ, *induced* by the vector field ξ. This formula shows, in particular, that the function $\Gamma_{\Omega,W}(\pi) \ni \gamma \longrightarrow (\partial_{J^r\xi}\lambda)_\Omega(\gamma) \in \mathbb{R}$ is the variational functional (over Ω) associated with the lagrangian $\partial_{J^r\xi}\lambda$. We call this function the *variational derivative*, or the *first variation* of λ_Ω by ξ.

We now compute the Lie derivative $\partial_{J^r\xi}\lambda$. Choose to this purpose a Lepage equivalent ρ of λ, and denote by s the *order* of ρ. Since $\lambda = h\rho$, or, which is the same, $J^r\gamma^*\lambda = J^s\gamma^*\rho$ for all sections γ, we obtain

$$J^r\gamma^*\partial_{J^r\xi}\lambda = J^s\gamma^*\partial_{J^s\xi}\rho = J^s\gamma^*(i_{J^s\xi}d\rho + di_{J^s\xi}\rho).$$

Omitting γ and using the Euler-Lagrange form (3), (4), we get

$$\partial_{J^r\xi}\lambda = hi_{J^{s+1}\xi}E_\lambda + hdi_{J^s\xi}\rho. \tag{9}$$

This is the *differential first variation formula*; the first term on the right is the *Euler-Lagrange term*, and the second one is the *boundary term*.

Writing (9) in coordinates, we obtain the well-known classical expressions, standing behind the variation integral.

2.6. Extremals

Let $\lambda \in \Omega^r_{n,X}W$ be a lagrangian, and let $\rho \in \Omega^s_n W$ be a Lepage equivalent of λ. We say that a section $\gamma \in \Gamma_{\Omega,W}(\pi)$ is *stable* with respect to a variation ξ of γ, if $(\partial_{J^r\xi}\lambda)_\Omega(\gamma) = 0$. Stable sections with respect to *families* of variations are defined in an obvious way. If γ is stable with respect to all ξ with support contained in $\pi^{-1}(\Omega)$, we say that γ is an *extremal* of λ_Ω. A section γ which is an extremal of every λ_Ω is called an *extremal* of λ.

The following conditions are equivalent: (1) γ is an extremal of λ, (2) γ satisfies

$$J^s\gamma^* i_{J^s\xi}d\rho = 0$$

for all π-vertical vector fields ξ, and (3) for every fibered chart on Y, γ satisfies the system of partial differential equations

$$E_\sigma(\mathcal{L}) \circ J^{2r}\gamma = 0.$$

2.7. Trivial lagrangians

A lagrangian $\lambda \in \Omega^r_{n,X}W$ is called *trivial* (or *variationally trivial*, or *null*) if there exists an $(n-1)$-form $\eta \in \Omega^s_{n-1}W$ such that $\lambda = hd\eta$. λ is called *locally trivial* if there exists an open covering $\{W_\iota\}_{\iota\in I}$ of Y, and to each $\iota \in I$ an $(n-1)$-form $\eta_\iota \in \Omega^s_{n-1}W_\iota$, such that $\lambda = hd\eta_\iota$ over W_ι.

The following is a standard consequence of variational sequence theory.

Theorem 1. *A lagrangian λ is locally trivial if and only if $E_\lambda = 0$.*

2.8. Locally variational forms

A 1-contact, $\pi^{s,0}$-horizontal form $\varepsilon \in \Omega^s_{n+1,Y}W$ is called a *dynamical form* (Krupková [22]; Takens [28] calls such forms *source forms*). From the definition it follows that in a fibered chart (V,ψ), $\psi = (x^i, y^\sigma)$,

$$\varepsilon = \varepsilon_\sigma\omega^\sigma \wedge \omega_0,$$

where $\varepsilon_\sigma = \varepsilon_\sigma(x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \dots, y^\sigma_{j_1 j_2 \dots j_s})$. We say that a dynamical form ε is *variational*, if $\varepsilon = E_\lambda$ for some lagrangian $\lambda \in \Omega^r_{n,X} W$. ε is said to be *locally variational*, if there are an open covering $\{V_\iota\}_{\iota \in I}$ of Y and a family $\{\lambda_\iota\}_{\iota \in I}$ of lagrangians $\lambda_\iota \in \Omega^r_{n,X} V_\iota$ such that for every $\iota \in I$,

$$\varepsilon|_{V_\iota} = E_{\lambda_\iota}.$$

Denote

$$H_\sigma^{j_1 j_2 \dots j_i}{}_\nu(\varepsilon) = \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{j_1 j_2 \dots j_i}} - (-1)^i \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{j_1 j_2 \dots j_i}} - \sum_{k=i+1}^{s} (-1)^k \binom{k}{i} d_{j_{i+1}} d_{j_{i+2}} \dots d_{j_k} \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{j_1 j_2 \dots j_i j_{i+1} \dots j_k}},$$

and

$$H_\varepsilon = \frac{1}{2} \sum_{i=1}^{s} H_\sigma^{j_1 j_2 \dots j_i}{}_\nu(\varepsilon) \omega^\sigma_{j_1 j_2 \dots j_i} \wedge \omega^\nu \wedge \omega_0.$$

The functions $H_\sigma^{j_1 j_2 \dots j_i}{}_\nu(\varepsilon)$, called the *Helmholtz expressions*, appeared for the first time in Aldersley [1]; H_ε is the (global) *Helmholtz form* (Anderson [2], Krupka [18], [20], Krbek and Musilová [14]).

The following is a consequence of the variational sequence theory.

Theorem 2. *A source form ε is locally variational if and only if $H_\varepsilon = 0$.*

2.9. Invariant transformations

An automorphism $\alpha : W \longrightarrow Y$ of the fibered manifold Y is said to be an *invariant transformation* of a form $\eta \in \Omega^s_p W$, if

$$J^s \alpha^* \eta = \eta.$$

We also say that η is *invariant* with respect to α. Let ξ be a π-projectable vector field on Y. We say that ξ is the *generator* of invariant transformations of η, if

$$\partial_{J^s \xi} \eta = 0.$$

In this case we also say that η is *invariant* with respect to ξ. These definitions include the notions of invariance of *lagrangians*, *dynamical forms*, and, in particular, the *Euler-Lagrange forms*.

Note that for *any* π-projectable vector field ξ, and any $\lambda \in \Omega^r_{n,X} W$,

$$\partial_{J^s \xi} E_\lambda = E_{\partial_{J^r \xi} \lambda}, \tag{10}$$

where s is the order of the Euler-Lagrange form E_λ. Thus, E_λ is invariant with respect to ξ if and only if $\partial_{J^r \xi} \lambda$ is a trivial lagrangian.

The following result is standard.

Theorem 3. *Let ξ be a π-projectable vector field on Y, and let $\lambda \in \Omega^{r+1}_{n,X} W$ be a lagrangian. The following conditions are equivalent:*

(a) *ξ generates invariant transformations of the Euler-Lagrange form E_λ.*

(b) *There exist an open covering $\{V_\iota\}_{\iota \in I}$ of Y and a system of $(n-1)$-forms $\{\eta_\iota\}_{\iota \in I}$, where $\eta_\iota \in \Omega^r_{n-1} V_\iota$, such that*

$$\partial_{J^{r+1} \xi} \lambda = h d \eta_\iota.$$

The following simple consequence of the first variation formula is known as the *Noether's theorem.*

Theorem 4. *Let $\lambda \in \Omega^{r+1}_{n,X}W$ be a lagrangian. Let $\rho \in \Omega^s_n W$ be a Lepage equivalent of λ, and let γ be an extremal.*

(a) *For any generator ξ of invariant transformations of λ,*

$$dJ^s\gamma^* i_{J^s\xi}\rho = 0.$$

(b) *For any generator ξ of invariant transformations of E_λ, there exist an open covering $\{V_\iota\}_{\iota\in I}$ of W and a family $\{\eta_\iota\}_{\iota\in I}$ of $(n-1)$-forms $\eta_\iota \in \Omega^r_{n-1}V_\iota$ such that for every $\iota \in I$,*

$$dJ^s\gamma^*(i_{J^s\xi}\rho - \eta_\iota) = 0.$$

3. Invariance: First order variational principles

One of specific features of the *first order* Lagrange structures consists in existence of two "simple" Lepage forms (Section 2.3). The first one is the *Poincaré-Cartan form*, whose order of contactness is ≤ 1 (see e.g. García [8], Goldschmidt and Sternberg [9], Krupka [15], Prieto [24]). The second one is the *fundamental Lepage form*, whose order of contactness is, in general, maximal, i.e., $\leq n$. We now compare invariance properties of these forms. Our results extend the usual concepts, based on the use of the Poincaré-Cartan form. For general approach to invariance we refer to Trautman [29] and Krupka [15], [19].

As before, we denote by Φ_λ the fundamental Lepage equivalent, associated with a first order lagrangian λ, and by Θ_λ the Poincaré-Cartan equivalent.

Theorem 5. *For any automorphism $\alpha : W \longrightarrow Y$ of Y,*

$$J^1\alpha^*\Phi_\lambda = \Phi_{J^1\alpha^*\lambda}. \tag{11}$$

Proof. 1. Let α_0 be the projection of α, and let (V,ψ), $\psi = (x^i, y^\sigma)$, and $(\bar{V},\bar{\psi})$, $\bar{\psi} = (\bar{x}^i, \bar{y}^\sigma)$, be two fibered charts such that $\alpha(V) \subset \bar{V}$. Let (U,φ), $\varphi = (x^i)$, and $(\bar{U},\bar{\varphi})$, $\bar{\varphi} = (\bar{x}^i)$ be the associated charts on X. Denote

$$\bar{x}^i\alpha_0\varphi^{-1} = f^i, \quad \bar{y}^\sigma\alpha\psi^{-1} = F^\sigma,$$

and

$$x^p\alpha_0^{-1}\bar{\varphi}^{-1} = g^p.$$

Clearly, on the corresponding domains,

$$f^i(g^1(\bar{x}^1,\bar{x}^2,\ldots,\bar{x}^n), g^2(\bar{x}^1,\bar{x}^2,\ldots,\bar{x}^n),\ldots, g^n(\bar{x}^1,\bar{x}^2,\ldots,\bar{x}^n)) = \bar{x}^i,$$
$$g^p(f^1(x^1,x^2,\ldots,x^n), f^2(x^1,x^2,\ldots,x^n),\ldots, f^n(x^1,x^2,\ldots,x^n)) = x^p.$$

From these formulas, we can easily derive equations of the mapping $J^1\alpha : W^1 \longrightarrow J^1Y$ in terms of the associated coordinates. By definition, we have for every $J^1_x\gamma \in W^1$, $J^1\alpha(J^1_x\gamma) = J^1_{\alpha_0(x)}(\alpha\gamma\alpha_0^{-1})$. On $V^1 \subset W^1$,

$$\bar{x}^iJ^1\alpha(J^1_x\gamma) = \bar{x}^iJ^1_{\alpha_0(x)}(\alpha\gamma\alpha_0^{-1}) = \bar{x}^i\alpha_0(x) = \bar{x}^i\alpha_0\varphi^{-1}(\varphi(x)),$$
$$\bar{y}^\sigma J^1\alpha(J^1_x\gamma) = \bar{y}^\sigma J^1_{\alpha_0(x)}(\alpha\gamma\alpha_0^{-1}) = \bar{y}^\sigma\alpha\psi^{-1}(\psi(\gamma(x))),$$

and

$$\bar{y}^\sigma_j J^1\alpha(J^1_x\gamma) = \bar{y}^\sigma_j J^1_{\alpha_0(x)}(\alpha\gamma\alpha_0^{-1}) = D_j(\bar{y}^\sigma\alpha\gamma\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x))).$$

Computing the derivative by the chain rule, we get

$$\begin{aligned} &D_j(\bar{y}^\sigma\alpha\gamma\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x))) \\ &\quad = D_{1,k}(\bar{y}^\sigma\alpha\psi^{-1})(\psi(\gamma(x)))D_j(x^k\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x))) \\ &\quad + D_{2,\nu}(\bar{y}^\sigma\alpha\psi^{-1})(\psi(\gamma(x)))D_k(y^\nu\gamma\varphi^{-1})(\varphi(x))D_j(x^k\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x))). \end{aligned}$$

We define functions $F^\sigma_j : V^1 \longrightarrow \mathbb{R}$ by

$$\begin{aligned} &F^\sigma_j(x^i(J^1_x\gamma), y^\tau(J^1_x\gamma), y^\tau_p(J^1_x\gamma)) \\ &\quad = D_{1,k}(\bar{y}^\sigma\alpha\psi^{-1})(\psi(\gamma(x)))D_j(x^k\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x))) \\ &\quad + D_{2,\nu}(\bar{y}^\sigma\alpha\psi^{-1})(\psi(\gamma(x)))D_k(y^\nu\gamma\varphi^{-1})(\varphi(x))D_j(x^k\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{\varphi}(\alpha_0(x))), \end{aligned}$$

or, which is the same, by

$$\begin{aligned} &F^\sigma_j(x^i, y^\tau, y^\tau_p) \\ &\quad = \left(\left(\frac{\partial F^\sigma}{\partial x^k}\right)_{(x^i,y^\tau)} + \left(\frac{\partial F^\sigma}{\partial y^\nu}\right)_{(x^i,y^\tau)} y^\nu_k\right)\left(\frac{\partial g^k}{\partial \bar{x}^j}\right)_{(f^1(x^i),f^2(x^i),\dots,f^n(x^i))} \\ &\quad = (d_kF^\sigma)(x^i, y^\tau, y^\tau_j)\left(\frac{\partial g^k}{\partial \bar{x}^j}\right)_{(f^1(x^i),f^2(x^i),\dots,f^n(x^i))}. \end{aligned}$$

Then

$$\bar{y}^\sigma_j J^1\alpha(\psi^1)^{-1} = F^\sigma_j.$$

Summarizing, we see that the mapping $J^1\alpha$ is expressed by equations

$$\begin{aligned} &\bar{x}^i\alpha_0\varphi^{-1} = f^i, \quad \bar{y}^\sigma\alpha\psi^{-1} = F^\sigma, \\ &\bar{y}^\sigma_j J^1\alpha(\psi^1)^{-1} = d_kF^\sigma \cdot \left(\frac{\partial g^k}{\partial \bar{x}^j} \circ \bar{\varphi}\alpha_0\varphi^{-1}\right). \end{aligned}$$

2. We now derive chart expressions for the forms $\alpha_0^*\bar{\omega}_0$ and $\alpha_0^*\bar{\omega}_{i_1i_2\dots i_k}$, where $1 \le k \le n$. We have, with obvious conventions,

$$\begin{aligned} \alpha_0^*\bar{\omega}_0(x) &= d(\bar{x}^1\alpha_0)(x) \wedge d(\bar{x}^2\alpha_0)(x) \wedge \dots \wedge d(\bar{x}^n\alpha_0)(x) \\ &= \det\left(\frac{\partial f^p}{\partial x^q}\right)_{\varphi(x)}\omega_0(x). \end{aligned}$$

Analogously, since

$$T_{\alpha_0(x)}\alpha_0^{-1} \cdot \left(\frac{\partial}{\partial \bar{x}^i}\right)_{\bar{\varphi}\alpha_0(x)} = \left(\frac{\partial g^j}{\partial \bar{x}^i}\right)_{\alpha_0(x)}\left(\frac{\partial}{\partial x^j}\right)_x,$$

and

$$\begin{aligned} &\alpha_0^*\bar{\omega}_i(x)(\xi_2, \xi_3, \dots, \xi_n) \\ &\quad = \left(\frac{\partial(x^j\alpha_0^{-1}\bar{\varphi}^{-1})}{\partial \bar{x}^i}\right)_{\bar{\varphi}\alpha_0(x)} \det\left(\frac{\partial(\bar{x}^p\alpha_0\varphi^{-1})}{\partial x^q}\right)_{\varphi(x)}\omega_j(x)(\xi_2, \xi_3, \dots, \xi_n), \end{aligned}$$

we have

$$\alpha_0^* \bar{\omega}_i(x) = \left(\frac{\partial g^j}{\partial \bar{x}^i}\right)_{\bar{\varphi}\alpha_0(x)} \det\left(\frac{\partial f^p}{\partial x^q}\right)_{\varphi(x)} \omega_j(x).$$

Continuing in the same way we obtain

$$\begin{aligned} &\alpha_0^* \bar{\omega}_{i_1 i_2 \dots i_k}(x)(\xi_{k+1}, \xi_{k+2}, \dots, \xi_n) \\ &\quad = \left(\frac{\partial g^{j_1}}{\partial \bar{x}^{i_1}}\right)_{\bar{\varphi}\alpha_0(x)} \left(\frac{\partial g^{j_2}}{\partial \bar{x}^{i_2}}\right)_{\bar{\varphi}\alpha_0(x)} \dots \left(\frac{\partial g^{j_k}}{\partial \bar{x}^{i_k}}\right)_{\bar{\varphi}\alpha_0(x)} \det\left(\frac{\partial f^p}{\partial x^q}\right)_{\varphi(x)} \\ &\quad \cdot \omega_{j_1 j_2 \dots j_k}(x)(\xi_{k+1}, \xi_{k+2}, \dots, \xi_n), \end{aligned}$$

i.e.,

$$\begin{aligned} \alpha_0^* \bar{\omega}_{i_1 i_2 \dots i_k}(x) &= \left(\frac{\partial g^{j_1}}{\partial \bar{x}^{i_1}}\right)_{\bar{\varphi}\alpha_0(x)} \left(\frac{\partial g^{j_2}}{\partial \bar{x}^{i_2}}\right)_{\bar{\varphi}\alpha_0(x)} \dots \left(\frac{\partial g^{j_k}}{\partial \bar{x}^{i_k}}\right)_{\bar{\varphi}\alpha_0(x)} \\ &\quad \cdot \det\left(\frac{\partial f^p}{\partial x^q}\right)_{\varphi(x)} \omega_{j_1 j_2 \dots j_k}(x). \end{aligned}$$

3. Similarly,

$$(J^1\alpha)^* \bar{\omega}^\sigma(J^1_x\gamma) = \left(\frac{\partial F^\sigma}{\partial y^\nu}\right)_{\psi\gamma(x)} \omega^\nu(J^1_x\gamma).$$

4. We now prove Theorem 5. To simplify our formulas, we sometimes write x, or $\gamma(x)$, instead of $J^1_x\gamma$. Let the lagrangian λ be expressed over $\bar{V}$ by

$$\lambda = \bar{\mathcal{L}} \bar{\omega}_0.$$

Then over V,

$$(J^1\alpha)^* \lambda(J^1_x\gamma) = (\bar{\mathcal{L}} \circ J^1\alpha(J^1_x\gamma)) \det\left(\frac{\partial f^p}{\partial x^q}\right)_{\varphi(x)} \omega_0(x).$$

We can express the form $\Phi_{J^1\alpha^*\lambda}$ over V. Taking into account the summand containing k exterior factors ω^σ, we have the form from formula (5),

$$\begin{aligned} (12) \qquad &\left(\frac{\partial^k(\bar{\mathcal{L}} \circ J^1\alpha \circ (\psi^1)^{-1})}{\partial y^{\sigma_1}_{j_1} \partial y^{\sigma_2}_{j_2} \dots \partial y^{\sigma_k}_{j_k}}\right)_{\psi^1(J^1_x\gamma)} \det\left(\frac{\partial f^p}{\partial x^q}\right)_{\varphi(x)} \\ &\cdot \omega^{\sigma_1}(J^1_x\gamma) \wedge \omega^{\sigma_2}(J^1_x\gamma) \wedge \dots \wedge \omega^{\sigma_k}(J^1_x\gamma) \wedge \omega_{j_1 j_2 \dots j_k}(J^1_x\gamma). \end{aligned}$$

But

$$\begin{aligned} &\left(\frac{\partial(\bar{\mathcal{L}} \circ J^1\alpha \circ (\psi^1)^{-1})}{\partial y^{\sigma_1}_{j_1}}\right)_{\psi^1(J^1_x\gamma)} \\ &\quad = \left(\frac{\partial \bar{\mathcal{L}}}{\partial \bar{y}^{\nu_1}_{p_1}}\right)_{\bar{\psi}^1 J^1\alpha(J^1_x\gamma)} \left(\frac{\partial F^{\nu_1}}{\partial y^{\sigma_1}}\right)_{\psi\gamma(x)} \left(\frac{\partial g^{j_1}}{\partial \bar{x}^{p_1}}\right)_{\bar{\varphi}\alpha_0(x)}, \end{aligned}$$

and in the same way

$$\left(\frac{\partial^k(\bar{\mathcal{L}}\circ J^1\alpha\circ(\psi^1)^{-1})}{\partial y^{\sigma_1}_{j_1}\partial y^{\sigma_2}_{j_2}\dots\partial y^{\sigma_k}_{j_k}}\right)_{\psi^1(J^1_x\gamma)}$$
$$=\left(\frac{\partial^k\bar{\mathcal{L}}}{\partial\bar{y}^{\nu_1}_{p_1}\partial\bar{y}^{\nu_2}_{p_2}\dots\partial\bar{y}^{\nu_k}_{p_k}}\right)_{\bar{\psi}^1J^1\alpha(J^1_x\gamma)}\left(\frac{\partial F^{\nu_1}}{\partial y^{\sigma_1}}\right)_{\psi\gamma(x)}\left(\frac{\partial g^{j_1}}{\partial\bar{x}^{p_1}}\right)_{\bar{\varphi}\alpha_0(x)}$$
$$\cdot\left(\frac{\partial F^{\nu_2}}{\partial y^{\sigma_2}}\right)_{\psi\gamma(x)}\left(\frac{\partial g^{j_2}}{\partial\bar{x}^{p_2}}\right)_{\bar{\varphi}\alpha_0(x)}\dots\left(\frac{\partial F^{\nu_k}}{\partial y^{\sigma_k}}\right)_{\psi\gamma(x)}\left(\frac{\partial g^{j_k}}{\partial\bar{x}^{p_k}}\right)_{\bar{\varphi}\alpha_0(x)}.$$

Consequently, (12) gives the expression

$$\begin{aligned}&\left(\frac{\partial^k\bar{\mathcal{L}}}{\partial\bar{y}^{\nu_1}_{p_1}\partial\bar{y}^{\nu_2}_{p_2}\dots\partial\bar{y}^{\nu_k}_{p_k}}\right)_{\bar{\psi}^1J^1\alpha(J^1_x\gamma)}\left(\frac{\partial F^{\nu_1}}{\partial y^{\sigma_1}}\right)_{\psi\gamma(x)}\left(\frac{\partial g^{j_1}}{\partial\bar{x}^{p_1}}\right)_{\bar{\varphi}\alpha_0(x)}\\&\cdot\left(\frac{\partial F^{\nu_2}}{\partial y^{\sigma_2}}\right)_{\psi\gamma(x)}\left(\frac{\partial g^{j_2}}{\partial\bar{x}^{p_2}}\right)_{\bar{\varphi}\alpha_0(x)}\dots\left(\frac{\partial F^{\nu_k}}{\partial y^{\sigma_k}}\right)_{\psi\gamma(x)}\left(\frac{\partial g^{j_k}}{\partial\bar{x}^{p_k}}\right)_{\bar{\varphi}\alpha_0(x)}\\&\cdot\det\left(\frac{\partial\bar{f}^p}{\partial x^q}\right)_{\varphi(x)}\\&\cdot\omega^{\sigma_1}(J^1_x\gamma)\wedge\omega^{\sigma_2}(J^1_x\gamma)\wedge\dots\wedge\omega^{\sigma_k}(J^1_x\gamma)\wedge\omega_{j_1j_2\dots j_k}(J^1_x\gamma).\end{aligned}\tag{13}$$

On the other hand, consider in Φ_λ the summand

$$\frac{\partial^k\bar{\mathcal{L}}}{\partial\bar{y}^{\sigma_1}_{j_1}\partial\bar{y}^{\sigma_2}_{j_2}\dots\partial\bar{y}^{\sigma_k}_{j_k}}\bar{\omega}^{\sigma_1}\wedge\bar{\omega}^{\sigma_2}\wedge\dots\wedge\bar{\omega}^{\sigma_k}\wedge\bar{\omega}_{j_1j_2\dots j_k}\tag{14}$$

over $\bar{V}$. Computing the pull-back $J^1\alpha^*\Phi_\lambda$, and in particular, the pull-back of the differential form (14), we obtain

$$\begin{aligned}&\left(\frac{\partial^k\bar{\mathcal{L}}}{\partial\bar{y}^{\sigma_1}_{j_1}\partial\bar{y}^{\sigma_2}_{j_2}\dots\partial\bar{y}^{\sigma_k}_{j_k}}\right)_{\bar{\psi}^1J^1\alpha(J^1_x\gamma)}\\&\cdot\left(\frac{\partial F^{\sigma_1}}{\partial y^{\nu_1}}\right)_{\psi\gamma(x)}\left(\frac{\partial F^{\sigma_2}}{\partial y^{\nu_2}}\right)_{\psi\gamma(x)}\dots\left(\frac{\partial F^{\sigma_k}}{\partial y^{\nu_k}}\right)_{\psi\gamma(x)}\\&\cdot\left(\frac{\partial g^{l_1}}{\partial\bar{x}^{j_1}}\right)_{\bar{\varphi}\alpha_0(x)}\left(\frac{\partial g^{l_2}}{\partial\bar{x}^{j_2}}\right)_{\bar{\varphi}\alpha_0(x)}\dots\left(\frac{\partial g^{l_k}}{\partial\bar{x}^{j_k}}\right)_{\bar{\varphi}\alpha_0(x)}\det\left(\frac{\partial\bar{f}^p}{\partial x^q}\right)_{\varphi(x)}\\&\cdot\omega^{\nu_1}(J^1_x\gamma)\wedge\omega^{\nu_2}(J^1_x\gamma)\wedge\dots\wedge\omega^{\nu_k}(J^1_x\gamma)\wedge\omega_{l_1l_2\dots l_k}(J^1_x\gamma).\end{aligned}\tag{15}$$

Since (13) and (15) agree, we are done.

Corollary 1. *For every π-projectable vector field ξ, the fundamental Lepage form Φ_λ satisfies*

$$\partial_{J^1\xi}\Phi_\lambda=\Phi_{\partial_{J^1\xi}\lambda}.$$

Corollary 2. *The Poincaré-Cartan form Θ_λ satisfies*

$$J^1\alpha^*\Theta_\lambda=\Theta_{J^1\alpha^*\lambda}\tag{16}$$

and

$$\partial_{J^1\xi}\Theta_\lambda = \Theta_{\partial_{J^1\xi}\lambda}. \tag{17}$$

Proof. From the properties of contact forms it follows that the forms of the same order of contactness on the left and right hand side of formula (11) agree. Formula (16) means just the equality of forms of order of contactness ≤ 1.

From Theorem 5 we can easily derive, for lagrangians of order 1, formula (10) of Section 2.9.

Corollary 3. *The Euler-Lagrange form E_λ satisfies*

$$\partial_{J^2\xi}E_\lambda = E_{\partial_{J^1\xi}\lambda}. \tag{18}$$

Proof. From Theorem 5 it follows that

$$\partial_{J^2\xi}p_1 d\Phi_\lambda = p_1\partial_{J^1\xi}d\Phi_\lambda = p_1 d\Phi_{\partial_{J^1\xi}\lambda},$$

which is exactly formula (18).

We are now in position to study symmetries of the first order Lagrange structures. According to the definition used by Prieto [24], an *infinitesimal symmetry* of a first order lagrangian λ is a vector field Ξ on J^1Y such that $\partial_\Xi\Theta_\lambda = -d\eta$ for some $(n-1)$-form η. Clearly, if Ξ is an infinitesimal symmetry, then $d\partial_\Xi\Theta_\lambda = 0$, and the converse holds locally. In the following theorem we consider infinitesimal symmetries of the form $\Xi = J^1\xi$, where ξ is a π-projectable vector field, and compare them with generators of invariant transformations of the Euler-Lagrange form.

Theorem 6. *Let λ be a first order lagrangian, and let ξ be a π-projectable vector field.*

(a) *ξ is the generator of invariant transformations of the Euler-Lagrange form E_λ if and only if $\partial_{J^1\xi}d\Phi_\lambda = 0$.*

(b) *If $\Xi = J^1\xi$ is an infinitesimal symmetry, then ξ generates invariant transformations of E_λ.*

Proof. (a) Suppose that $\partial_{J^2\xi}E_\lambda = 0$. Then from Corollary 3, $E_{\partial_{J^1\xi}\lambda} = 0$, hence $d\Phi_{\partial_{J^1\xi}\lambda} = 0$ and according to Theorem 5, $\partial_{J^1\xi}d\Phi_\lambda = 0$. The converse is proved by reversing the arguments.

(b) Supposing that $d\partial_{J^1\xi}d\Theta_\lambda = 0$ we obtain $d\Theta_{\partial_{J^1\xi}\lambda} = 0$ (Corollary 2) and by definition,

$$p_1 d\Theta_{\partial_{J^1\xi}\lambda} = E_{\partial_{J^1\xi}\lambda} = \partial_{J^2\xi}E_\lambda = 0.$$

Remark 1. In Theorem 6, we give some properties of generators of invariant transformations of the Euler-Lagrange form on one side, and infinitesimal symmetries on the other side. Note that for several reasons, the definition of infinitesimal symmetry in its full generality does not seem well-motivated. First, variations, induced by general vector fields on J^1Y do not transform sections of the fibered manifold Y into sections of Y; in particular, such variations do not transform solutions of the Euler-Lagrange equations into solutions. Second, according to Theorem 6, infinitesimal symmetries do not include all generators of invariant transformations of the Euler-Lagrange form. The third reason consists in impossibility to generalize the definition of an infinitesimal symmetry to r-th order Lagrange structures, because for lagrangians of order $r \geq 3$ we do not have a global analogue of the Poincaré-Cartan form. For these reasons, we prefer, in the theory of local variational principles presented below, the concept of a generator of invariant transformations of the Euler-Lagrange form.

Remark 2. It is not known whether there exists a generalization of the fundamental Lepage form Φ_λ to higher order Lagrange structures.

4. Local variational principles

4.1. Local variational principles

Let $\varepsilon \in \Omega^s_{n+1,Y}Y$ be a locally variational form (ε is supposed to be defined globally). According to Section 2.8, the fibered manifold Y can be covered by open sets V_ι, $\iota \in I$, such that to every ι, there exists a lagrangian λ_ι over V_ι for the form $\varepsilon|_{V_\iota}$; over the intersections $V_\iota \cap V_\kappa$, the lagrangians λ_ι and λ_κ differ by a trivial lagrangian. In general, a globally defined lagrangian for ε need not exist.

In our definition of a local variational principle, we rephrase these properties of locally variational forms in terms of the Lepage forms. We say that a family $\{(V_\iota, \rho_\iota)\}_{\iota\in I}$, in which $\{V_\iota\}_{\iota\in I}$ is an open covering of Y and for every $\iota \in I$, $\rho_\iota \in \Omega^s_n V_\iota$ is a Lepage form, is said to be a *local variational principle*, if for every $\iota, \kappa \in I$,

$$p_1 d\rho_\iota = p_1 d\rho_\kappa$$

over $V_\iota \cap V_\kappa$. The integer s is called the *order* of the local variational principle $\{(V_\iota, \rho_\iota)\}_{\iota\in I}$.

Suppose that we have a local variational principle $\{(V_\iota, \rho_\iota)\}_{\iota\in I}$ of order s. For every $\iota \in I$, we denote

$$E_\iota = p_1 d\rho_\iota.$$

E_ι is the Euler-Lagrange form of the *associated lagrangian* $\lambda_\iota = h\rho_\iota$, defined over V_ι. Since by definition, $E_\iota = E_\kappa$ for all $\iota, \kappa \in I$, setting

$$E = E_\iota$$

over V_ι, we obtain a global differential form E on $J^{s+1}Y$. This form is called the *Euler-Lagrange form*, associated with the local variational principle $\{(V_\iota, \rho_\iota)\}_{\iota\in I}$. Obviously, the Euler-Lagrange form is dynamical, locally variational form; it is not necessarily (globally) variational.

A local variational principle in another geometric context (i.e., on manifolds without fibration) was formulated by Dedecker [5]. Our definition is close to the Dedecker's approach.

Two local variational principles $\{(V_\iota, \rho_\iota)\}_{\iota\in I}$, $\{(V'_\kappa, \rho'_\kappa)\}_{\kappa\in K}$ are *equivalent*, if the associated Euler-Lagrange forms E, E' coincide, i.e., $E = E'$.

Theorem 7. *A family $\{(V_\iota, \rho_\iota)\}_{\iota\in I}$, in which $\{V_\iota\}_{\iota\in I}$ is an open covering of Y and for every $\iota \in I$, $\rho_\iota \in \Omega^s_n V_\iota$ is a Lepage form, is a local variational principle if and only if to every $\iota, \kappa \in I$, there exists a form $\eta_{\iota\kappa} \in \Omega^r_{n-1}(V_\iota \cap V_\kappa)$ and a contact form $\chi_{\iota\kappa} \in \Omega^r_n(V_\iota \cap V_\kappa)$ such that over $V_\iota \cap V_\kappa$,*

$$\rho_\iota - \rho_\kappa = d\eta_{\iota\kappa} + \chi_{\iota\kappa}. \tag{19}$$

Proof. If $\rho_\iota - \rho_\kappa = d\eta_{\iota\kappa} + \chi_{\iota\kappa}$, for some $\eta_{\iota\kappa}$ and $\chi_{\iota\kappa}$, then $d(\rho_\iota - \rho_\kappa) = d\chi_{\iota\kappa}$. This means that the class of $\chi_{\iota\kappa}$ is a contact Lepage form. Since $p_1 d\chi_{\iota\kappa}$ depends on the corresponding lagrangian only, that is, on $h\chi_{\iota\kappa}$ (see Section 2.3), and this lagrangian is zero, we have $p_1 d\chi_{\iota\kappa} = 0$. Consequently, $p_1 d\rho_\iota = p_1 d\rho_\kappa$. Conversely, if $p_1 d\rho_\iota = p_1 d\rho_\kappa$, then the Euler-Lagrange form $E_{h(\rho_\iota - \rho_\kappa)}$ vanishes. This means that the lagrangian $h(\rho_\iota - \rho_\kappa)$ is trivial, which implies (19).

4.2. First variation formula, extremals

A basic tool for an analysis of extremals and invariant transformations of a variational functional is the first variation formula. We now give a formulation of the first variation formula for local variational principles.

Let $\{(V_\iota, \rho_\iota)\}_{\iota \in I}$ be a local variational principle of order s. Fix an index $\iota \in I$, and choose a piece $\Omega \subset \pi(V_\iota)$. Then we have the variational functional

$$\Gamma_{\Omega, V_\iota}(\pi) \ni \gamma \longrightarrow \rho_{\iota,\Omega}(\gamma) = \int_\Omega J^s\gamma^*\rho_\iota \in \mathbb{R}.$$

For any π-projectable vector field ξ on Y, we have the *first variation formula*

$$\partial_{J^s\xi}\rho_\iota = i_{J^s\xi}d\rho_\iota + di_{J^s\xi}\rho_\iota.$$

This formula can easily be written by means of the associated lagrangian $\lambda_\iota = h\rho_\iota$. Since $\partial_{J^{s+1}\xi}h\rho_\iota = h\partial_{J^s\xi}\rho_\iota = hi_{J^s\xi}d\rho_\iota + hdi_{J^s\xi}\rho_\iota$, we have

$$\partial_{J^{s+1}\xi}h\rho_\iota = hi_{J^{s+1}\xi}p_1 d\rho_\iota + hdi_{J^s\xi}\rho_\iota = hi_{J^{s+1}\xi}E + hdi_{J^s\xi}\rho_\iota,$$

and

$$\partial_{J^{s+1}\xi}\lambda_\iota = hi_{J^{s+1}\xi}E + hdi_{J^s\xi}\rho_\iota,$$

where E is the Euler-Lagrange form of $\{(V_\iota, \rho_\iota)\}_{\iota \in I}$. This is another formulation of the first variation formula for the local variational principle $\{(V_\iota, \rho_\iota)\}_{\iota \in I}$.

We have the following simple observation.

Theorem 8. *Let $\{(V_\iota, \rho_\iota)\}_{\iota \in I}$ be a local variational principle of order s. Let γ be a section of Y. The following conditions are equivalent:*

(a) *For every $\iota \in I$, $\gamma_\iota = \gamma|_{\pi(V_\iota)}$ is an extremal of the variational functional $\rho_{\iota,\Omega}$.*

(b) *For every π-projectable vector field ξ, γ satisfies*

$$J^{s+1}\gamma^* i_{J^{s+1}\xi}E = 0.$$

A section γ, satisfying any of these two equivalent conditions, is called an *extremal* of the local variational principle $\{(V_\iota, \rho_\iota)\}_{\iota \in I}$.

4.3. Invariant transformations

It is straightforward to extend the theory of invariant transformations as introduced in Section 2.9, to *local variational principles.* The concept of a lagrangian in this case is defined only locally, but we still have the notions of invariance of the Euler-Lagrange form.

Suppose that we have a local variational principle $\{(V_\iota, \rho_\iota)\}_{\iota \in I}$ of order s, and denote by E its Euler-Lagrange form. Let $\alpha : W \longrightarrow Y$ be an automorphism of Y. We say that α is an *invariant transformation* of E, if

$$J^{s+1}\alpha^*E = E.$$

A π-projectable vector field ξ on Y is said to be the *generator of invariant transformations* of E, if

$$\partial_{J^{s+1}\xi} E = 0.$$

The following is straightforward.

Theorem 9. *Let* $\{(V_\iota, \rho_\iota)\}_{\iota \in I}$ *be a local variational principle, and let* ξ *be a* π*-projectable vector field. Let* E *be the Euler-Lagrange form of* $\{(V_\iota, \rho_\iota)\}_{\iota \in I}$*. The following conditions are equivalent:*

(a) ξ *is a generator of invariant transformations of* E*.*

(b) *There exists a family* $\{\eta_\iota\}_{\iota \in I}$ *of* $(n-1)$*-forms* $\eta_\iota \in \Omega^s_{n-1} V_\iota$ *such that for every* $\iota \in I$*,*

(20) $$h i_{J^{s+1}\xi} E + h d(i_{J^s\xi}\rho_\iota - \eta_\iota) = 0.$$

Proof. Let ξ be a generator of invariant transformations of E, let $\iota \in I$. Over V_ι, $E = E_{\lambda_\iota}$, where $\lambda_\iota = h\rho_\iota$, and $\partial_{J^{s+1}\xi} E = E_{\partial_{J^{s+1}\xi}\lambda_\iota} = 0$, hence by Theorem 3, $\partial_{J^{s+1}\xi}\lambda_\iota = h d\eta_\iota$ for some $(n-1)$-form η_ι over V_ι. Then

$$\partial_{J^{s+1}\xi}\lambda_\iota = h i_{J^{s+1}\xi} E + h d i_{J^s\xi}\rho_\iota = h d\eta_\iota,$$

proving (20).

Consider the Euler-Lagrange form E of the local variational principle $\{(V_\iota, \rho_\iota)\}_{\iota \in I}$, and a vector field ξ on Y. Let (V, ψ), $\psi = (x^i, y^\sigma)$, be a fibered chart on Y such that $V \subset V_\iota$. Suppose that over V

$$h\rho_\iota = \mathcal{L}_\iota \omega_0,$$

and

$$\xi = \xi_0^k \frac{\partial}{\partial x^k} + \xi^\sigma \frac{\partial}{\partial y^\sigma}.$$

Then over V,

$$E = E_\sigma(\mathcal{L}_\iota)\omega^\sigma \wedge \omega_0,$$

where

$$E_\sigma(\mathcal{L}_\iota) = \sum_{k=0}^{r} (-1)^k d_{i_1} d_{i_2} \dots d_{i_k} \frac{\partial \mathcal{L}_\iota}{\partial y^\sigma_{i_1 i_2 \dots i_k}},$$

and

(21) $$h i_{J^{s+1}\xi} E = E_\sigma(\mathcal{L}_\iota)(\xi^\sigma - y^\sigma_k \xi_0^k)\omega_0.$$

Formula (21) shows that the Euler-Lagrange equations for extremals are, over V,

(22) $$E_\sigma(\mathcal{L}_\iota) = 0.$$

Thus, if ξ generates invariant transformations of E, we have a *conservation law*

(23) $$d(i_{J^s\xi}\rho_\iota - \eta_\iota) = 0$$

(along any extremal). The arising equations (23) should be considered together with equations (22).

The set of generators of invariant transformations of the Euler-Lagrange form is a Lie algebra. Indeed, if two π-projectable fields ξ and ζ, satisfy

$$\partial_{J^s\xi} E = 0, \quad \partial_{J^s\zeta} E = 0,$$

then since $J^s[\xi,\zeta] = [J^s\xi, J^s\zeta]$, we have

$$\partial_{J^s[\xi,\zeta]}E = \partial_{J^s\xi}\partial_{J^s\zeta}E - \partial_{J^s\zeta}\partial_{J^s\xi}E = 0.$$

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