Some Recent Developments in Symmetries and Conservation Laws for Partial Differential Equations

This course will be concerned with recent developments on how to find and use symmetries for PDEs; how to find and use conservation laws for PDEs; connections between symmetries and conservation laws for PDEs. Much of the material in this course will appear in a forthcoming book to be published in the Springer Applied Mathematical Sciences series. Background material appears in *Symmetry and Integration Methods for Differential Equations* (Springer 2002) by George W. Bluman and Stephen C. Anco, which focused on Lie groups of transformations and their applications to solving ordinary differential equations (ODEs), the construction of conservation laws (integrating factors) for ODEs, and finding invariant solutions of PDEs.

Topics to be selected from:

1. Local symmetries—point, contact, higher-order.
2. How to find local symmetries admitted by a given PDE system.
3. Noether's theorem and its limitations.
4. How to directly find local conservation laws of a given PDE system.
5. Connections between symmetries and conservation laws.
6. Linearization of PDEs by invertible mappings through admitted symmetries; through admitted conservation law multipliers.
7. How to systematically find trees of equivalent but nonlocally related PDE systems for a given PDE system.
8. How to systematically find nonlocal symmetries and nonlocal conservation laws for a given PDE system.

Overview of the subject matter of this course.

In the latter part of the 19th century Sophus Lie initiated his studies on continuous groups (Lie groups) in order to put order to, and thereby extend systematically, the hodgepodge of heuristic techniques for solving ODEs. He showed that the problem of finding the Lie group of point transformations (point symmetries) admitted by a DE (ordinary or partial) reduced to solving related linear systems of determining equations for its infinitesimal generators. Lie also showed that an admitted point symmetry led to reducing the order of an ODE (irrespective of any imposed initial conditions) and, in the case of a PDE, to finding special solutions called invariant (similarity) solutions. Moreover, he showed that an admitted point symmetry generates a one-parameter family of solutions from a known solution of a DE. Most importantly, the applicability of Lie's work extends to nonlinear DEs. This work is discussed in the above-mentioned book as well as many other excellent references therein. The direct applicability of Lie's work to PDEs (especially nonlinear PDEs) is rather limited, even when a given PDE admits a point symmetry, since the resulting invariant solutions yield only a small subset of the solution set of the PDE and hence few posed boundary value problems can be solved.

The extensions of Lie's work to PDEs have focused on how to find and implement further applications for admitted symmetries, how to extend the applications arising from admitted symmetries to applications arising from generalizations of symmetries (e.g., conservation laws arising from multipliers, solutions arising from the “nonclassical method”), how to extend the spaces of admitted symmetries, and how to efficiently solve the (overdetermined) linear system of symmetry determining equations through the development of symbolic computation software as well as related calculations to find multipliers for conservation laws or to solve the nonlinear system of determining equations for the nonclassical method.

A symmetry of a PDE is any transformation of its solution manifold into itself, i.e., a symmetry transforms (maps) any solution of a PDE to another solution of the same PDE. Consequently, continuous symmetry transformations are defined topologically and are not restricted to admitted point symmetries. Thus, in principle, any nontrivial PDE admits symmetries. The problem is to find and use admitted symmetries. Practically, to find an admitted symmetry one must consider transformations, acting on spaces of a finite number of variables, which leave invariant the solution manifold of the given PDE and its differential consequences.

One such extension is to consider *higher-order symmetries* (*local symmetries*) where the solutions of the determining equations for symmetries are allowed to depend on a finite number of derivatives of the given dependent variables of the PDE (point symmetries depend at most linearly on the first derivatives of the dependent variables and contact symmetries allow dependence at most on first derivatives). In making this extension, it is essential to realize that the linear determining equations for local symmetries are the linearized system of the given PDE that holds for *all* of its solutions. Globally, point and contact symmetries act on finite-dimensional spaces whereas higher-order symmetries act on infinite-dimensional spaces consisting of the dependent and independent variables as well as *all* of their derivatives. Well-known integrable equations of mathematical physics such as the Korteweg-de-Vries equation admit an infinite number of higher-order symmetries.

Another extension is to consider solutions of the determining equations that allow an ad-hoc dependence on nonlocal variables such as integrals of the dependent variables. Usually such symmetries are found formally through recursion operators that depend on inverse differentiation. Integrable equations such as the sine-Gordon and cubic Schrodinger equations admit an infinite number of such symmetries.

In her celebrated 1918 paper, Emmy Noether showed that if a system of DEs admits a variational principle, then any local transformation group leaving invariant the action integral for its Lagrangian density, i.e., an admitted *variational symmetry*, yields a conservation law. Conversely, any conservation law admitted by a variational system of DEs arises from a variational symmetry, and hence there is a direct correspondence between conservation laws and variational symmetries (Noether’s theorem).

There are several limitations to Noether’s theorem for finding conservation laws for a given system of DEs. First of all, it is restricted to variational systems. Consequently, for this theorem to be applicable to a given system as written, the system must be of even order, have the same number of dependent variables as the number of equations in the given system, and have no dissipation. In particular, a given system of DEs is variational if and only if its linearized system is self-adjoint. There is also the difficulty of finding symmetries admitted by the action integral. In general, not all admitted local symmetries of a variational system of DEs are variational symmetries. Moreover, the use of Noether’s theorem to find conservation laws is coordinate dependent.

A conservation law of a given system of PDEs is a divergence expression that vanishes on all solutions of the PDEs. In general all such divergences arise from a scalar product formed by multipliers, depending on any independent and dependent variables as well as at most a finite number of derivatives of the dependent variables of the system, with each PDE in the system. It then follows that a given system of PDEs has a conservation law if and only if there exist multipliers whose scalar product with the PDEs in the system is identically annihilated by the Euler operators associated with each of its dependent variables without restricting these variables in the scalar product to solutions of the system of PDEs. If a given PDE system is variational then its multipliers are variational symmetries. In this case, it turns out that all multipliers satisfy the linearized PDE system augmented by additional determining equations that correspond to the action integral being invariant under the associated variational symmetry. In general, all multipliers are the solutions of a linear determining system that includes the adjoint system of the linearized PDE system. For any multiplier yielding a conservation law, there is an integral formula that yields the fluxes and densities of any admitted conservation law without need of a specific Lagrangian even in the case when the given system of DEs is variational.

Another important application of symmetries to PDEs is to determine whether or not a given PDE can be mapped into an equivalent target PDE of interest. This is especially significant if a target class of PDEs can be completely characterized in terms of admitted symmetries. Target classes which such characterizations include linear systems and linear PDEs with constant coefficients. Consequently, from the admitted point or contact symmetries of a given system of PDEs, one can determine whether or not it can be mapped into a linear PDE by a point or contact transformation and find such an explicit mapping when one exists. Moreover, one can also see whether or not such a linearization is possible from its admitted multipliers for conservation laws. From the admitted point symmetries of a linear PDE with variable coefficients, one can determine whether or not it can be mapped by a point transformation into a linear PDE with constant coefficients and find such an explicit mapping when one exists.

In order to effectively apply symmetry methods to PDEs, one needs to work in some coordinate frame in order to perform calculations. A systematic procedure to find symmetries that are nonlocal and yet are local in some related coordinate frame involves embedding a given PDE system in another PDE system obtained by adjoining nonlocal variables in such a way that the related PDE system is equivalent to the given PDE system and the given PDE system arises through projection. Consequently, any local symmetry of the related system yields a symmetry of the given system. If the local symmetry of the related PDE system has an essential dependence on the nonlocal variables after projection, then it yields a nonlocal symmetry of the given PDE system.

A systematic way to find such an embedding is through conservation laws of a given PDE system of. For each conservation law, one can introduce potentials. By adjoining the resulting potential equations to the given system, one can construct an augmented PDE system (*potential system*). By construction, such a potential system is nonlocally equivalent to the given PDE system since, through built in integrability conditions, any solution of the given PDE system yields a solution of the potential system and, conversely, through projection any solution of the potential system yields a solution of the given PDE system. But this relationship is nonlocal since there is not a one-to-one correspondence between solutions of the given and potential systems. If a local symmetry of the potential system depends essentially on the potential variables when projected to the given PDE system, then it yields a nonlocal symmetry (*potential symmetry*) of the given system. It turns out that many PDE systems admit such potential symmetries. Moreover, one can find further nonlocal symmetries of the given PDE system through seeking local symmetries of equivalent subsystems of the given system or potential system provided such subsystems are nonlocally equivalent to the given PDE system. Invariant solutions of such potential systems and subsystems can yield further solutions of the given PDE system. Since a potential symmetry is a local symmetry of a potential system, it generates a one-parameter family of solutions from any known solution of the potential system that in turn yields a one-parameter family of solutions from a known solution of the given PDE system. Furthermore, conservation laws of potential systems (and subsystems) can yield nonlocal conservation laws of given systems. Linearizations of such potential systems through local symmetry or conservation law analysis can yield explicit nonlocal linearizations of given systems of PDEs. Moreover, through a potential system one can extend the mappings of linear systems with variable coefficients to linear systems with constant coefficients to include nonlocal mappings between such systems.

One can further extend embeddings through using conservation laws to systematically construct trees of nonlocally related but equivalent systems of PDEs. If a given PDE system has n local conservation laws, then each conservation law yields potentials and a corresponding potential system. Most importantly, from the n conservation laws, one can directly construct $2^n - 1$ independent nonlocally related systems of PDEs by considering the corresponding potential systems individually (n singlets), in pairs ($n(n-1)/2$ couplets),…, taken all together (one n-plet). In turn, any one of these $2^n - 1$ systems could lead to the discovery of new nonlocal symmetries and/or nonlocal conservation laws of the given PDE system or any of the other 2^n nonlocally related systems. Moreover, such nonlocal conservation laws could yield further nonlocally related systems, etc. Furthermore, subsystems of such nonlocally related systems could yield further nonlocally related systems. Correspondingly, a tree of nonlocally related systems is constructed. Through such constructions, one can systematically relate Eulerian and Lagrangian coordinate descriptions as well as find other descriptions of gas dynamics and nonlinear elasticity equations. In both cases, subsystems of potential systems arising from their systems written in Eulerian coordinates yield the corresponding systems in Lagrangian coordinates.

For a given class of PDEs with constitutive functions, it is of interest to classify its trees of nonlocally related systems and corresponding symmetries and conservation laws with respect to various forms of its constitutive functions. When a system is variational, i.e., its linearized system is self-adjoint, then of course the conservation laws arise from a subset of its symmetries and, in particular, the number of linearly independent conservation laws cannot exceed the number of higher-order symmetries. But from the above, we see that this will not be the case when a system is not variational. Here a constitutive function could yield more conservation laws than symmetries as well as vice versa.

For any given system of PDEs, an admitted symmetry (continuous or discrete) yields a formula that maps a conservation law to a conservation law of the same system, whether or not the given system is variational. If the symmetry is continuous, then in terms of a parameter expansion a given conservation law could map into more than one new conservation law for the given system.

Another important extension relates to Lie's work on finding invariant solutions for PDEs. As mentioned previously, a point symmetry admitted by a PDE maps each of its solutions into a one-parameter family of solutions. But some solutions map into themselves, i.e., they are themselves invariant. Such solutions satisfy the characteristic PDE that is the invariant surface condition yielding the invariants for the point symmetry. The invariant solutions arising from the point symmetry are the solutions of the given PDE that satisfy the augmented system consisting of its characteristic PDE with known coefficients (obtained from the point symmetry) and the given PDE itself. The invariant solutions arise as solutions of a reduced system with one less independent variable. This method ("classical method") of Lie to find invariant solutions of a given PDE generalizes to the *nonclassical method* introduced in 1967 where one seeks solutions of an augmented system consisting of the given PDE and the characteristic PDE with unknown coefficients as well as differential consequences of the characteristic PDE. Here the unknown coefficients are determined by substituting the characteristic equation into the determining system for symmetries of the augmented system. The resulting over-determined system is nonlinear (even if the given PDE is linear) in these unknown coefficients, but less over-determined than is the case when finding point symmetries of the given PDE. Each solution of the determining system for point symmetries is a solution of the determining system for the unknown coefficients of the characteristic PDE. Solving for the unknown coefficients, one then proceeds to find the corresponding "nonclassical" solutions of the augmented system that, by construction, include the classical invariant solutions.

The solutions of a PDE that can be obtained by the nonclassical method include all of its solutions that satisfy a particular functional form (ansatz) of some generality that allows an arbitrary dependence on a similarity variable (depending on the independent and dependent variables of the PDE) and an arbitrary dependence on a function of a similarity variable and the independent variables of the PDE. The solutions obtained by the nonclassical method include all solutions obtained "directly" from such an ansatz by the *direct method* introduced by Clarkson and Kruskal in 1988.